

Handout 9: Dynamic Inventory Control

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Suggested Reading: Section 4.2 of Bertsekas, *Dynamic Programming and Optimal Control: Volume I (3rd Edition)*, Athena Scientific, 2005.

1 Inventory control without setup cost: order-up-to policy

Consider again the inventory control problem. We assume that excess demand at each period is backlogged and is filled when additional inventory becomes available. This is represented by negative inventory in the system equation

$$x_{k+1} = x_k + u_k - w_k, k = 0, 1, \dots, N-1,$$

where x_k (might be negative) denotes the inventory level at the beginning of period k , u_k denotes the quantity to order in period k , and w_k denotes the random demand in period k . Note that when $x_k < 0$, then $-x_k$ indicates the backlog, the number of units of unmet demand in the previous period. We assume the holding/shortage cost takes the form:

$$r(x) = p \max(0, -x) + h \max(0, x),$$

where p and h are given nonnegative scalars. Thus the total expected cost to be minimized is

$$E \left\{ \sum_{k=0}^{N-1} (cu_k + r(x_k + u_k - w_k)) \right\},$$

i.e.,

$$E \left\{ \sum_{k=0}^{N-1} (cu_k + p \max(0, -(x_k + u_k - w_k)) + h \max(0, x_k + u_k - w_k)) \right\},$$

where c is the per unit purchasing cost.

By applying the DP algorithm, we have

$$J_N(x_N) = 0,$$

$$J_k(x_k) = \min_{u_k \geq 0} [cu_k + H(x_k + u_k) + E \{J_{k+1}(x_k + u_k - w_k)\}], \quad (1)$$

where the function H is defined by

$$H(y) = E \{r(y - w_k)\} = pE \{\max(0, w_k - y)\} + hE \{\max(0, y - w_k)\}.$$

Note that H depends on k whenever the probability distribution of w_k depends on k . Also note that H is convex, since $r(y - w_k)$ is convex in y for each fixed w_k , and taking expectation over w_k preserves convexity.

By introducing the variable $y_k = x_k + u_k$, we can write the DP Eq. (1) as

$$J_k(x_k) = \min_{y_k \geq x_k} G_k(y_k) - cx_k, \quad (2)$$

where

$$G_k(y) = cy + H(y) + E\{J_{k+1}(y - w)\}. \quad (3)$$

We will prove shortly that the function G_k is convex, but for the moment let us assume that it is convex. Suppose that G_k has an unconstrained minimum with respect to y , denoted by S_k :

$$S_k := \arg \min_{y \in \mathbb{R}} G_k(y).$$

Then, in view of the constraint $y_k \geq x_k$, it is seen that a minimizing y_k in Eq. (2) equals S_k if $x_k < S_k$, and equals x_k otherwise. Since $u_k = y_k - x_k$, we see that the minimum in the DP equation (1) is attained at $u_k = S_k - x_k$ if $x_k < S_k$, and at $u_k = 0$ otherwise. An optimal policy is determined by the sequence of scalars $\{S_0, S_1, \dots, S_{N-1}\}$ and has the form

$$\mu_k^*(x_k) = \begin{cases} S_k - x_k & \text{if } x_k < S_k, \\ 0 & \text{if } x_k \geq S_k. \end{cases} \quad (4)$$

Thus, the optimality of the policy (4) will be proved if we can show that the cost-to-go functions J_k (and hence also the functions G_k of Eq. (3)) are convex, and furthermore $\lim_{|y| \rightarrow \infty} G_k(y) = \infty$, so that the minimizing scalars S_k exist. This can be proved inductively.

Two key features: Preservation and Attainment.

1.1 Convex function.

Definition. A given function $f : \mathbb{R} \rightarrow \mathbb{R}$ is convex if and only if for any x, y , the following inequality holds for any scalar $\lambda \in (0, 1)$:

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

f is called concave if $-f$ is convex.

- First order condition characterization: Suppose f is differentiable. Then, f is convex if and only if $f(y) \geq f(x) + \nabla f(x) \cdot (y - x)$.
- Second order condition characterization: Suppose f is twice differentiable, then f is convex if and only if the Hessian matrix $\nabla^2 f \succeq 0$.

Examples of convex functions.

- $f(x) = x^2$ is convex
- $f(x) = x$ is both convex and concave
- $f(x) = |x|$ is convex
- $f(x) = x^3$ is convex on $[0, \infty)$ and concave on $(-\infty, 0]$

Preservation of Convexity.

- $\sum_{i=1}^m w_i f_i(x)$ is convex if $w_i \geq 0$ and f_i is convex, for all i .
- $\int_{\mathcal{A}} w(y) f(x, y) dy$ is convex if $w(\cdot) \geq 0$ and f is convex in x , for any y .
- $E[f(x, Y)]$ is convex in x if $f(x, y)$ is convex in x , for any y .
- $\max\{f_1(x), \dots, f_m(x)\}$ is convex if each f_i is convex.
- $\min_{y \in C} f(x, y)$ is convex if f is convex in (x, y) and C is a convex nonempty set.

2 Inventory control with positive fixed cost: (s, S) policy

We now turn to the more complicated case where there is a positive fixed cost K associated with a positive inventory order. Thus the cost for ordering inventory $u \geq 0$ is

$$C(u) = \begin{cases} K + cu & \text{if } u > 0, \\ 0 & \text{if } u = 0. \end{cases}$$

The DP algorithm takes the form

$$J_N(x_N) = 0, \\ J_k(x_k) = \min_{u_k \geq 0} [C(u_k) + H(x_k + u_k) + E\{J_{k+1}(x_k + u_k - w_k)\}],$$

with H defined as earlier by

$$H(y) = E\{r(y - w)\} = pE\{\max(0, w - y)\} + hE\{\max(0, y - w)\}.$$

Consider again the functions

$$G_k(y) = cy + H(y) + E\{J_{k+1}(y - w)\}.$$

Then J_k is written as

$$J_k(x_k) = \min \left[G_k(x_k), \min_{u_k > 0} [K + G_k(x_k + u_k)] \right] - cx_k,$$

or equivalently, through the change of variable $y_k = x_k + u_k$,

$$J_k(x_k) = \min[G_k(x_k), \min_{y_k > x_k} [K + G_k(y_k)]] - cx_k.$$

Unfortunately, when $K > 0$ it is not necessarily true that the functions G_k are convex. However, it can be proved that all functions G_k are K -convex. And it can be verified that the policy

$$\mu_k^*(x_k) = \begin{cases} S_k - x_k & \text{if } x_k < s_k \\ 0 & \text{if } x_k \geq s_k, \end{cases} \quad (5)$$

is optimal, where S_k is a value of y that minimizes $G_k(y)$ and s_k is the smallest value of y for which $G_k(y) = K + G_k(S_k)$. A policy of the form (5) is known as a multi-period (s, S) policy.

2.1 K -convex function.

Definition. A given function $f : \mathfrak{R} \rightarrow \mathfrak{R}$ is K -convex if and only if for any x, y , the following inequality holds for any scalar $\lambda \in (0, 1)$:

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)(K + f(y)).$$

An equivalent definition. We say that a real-valued function f is K -convex, where $K \geq 0$, if

$$K + f(z + y) \geq f(y) + z \left(\frac{f(y) - f(y - b)}{b} \right), \text{ for all } z \geq 0, b > 0, y.$$

- For a function f to be K -convex, it must lie below the line segment connecting $(x, f(x))$ and $(y, K + f(y))$, for all real numbers x and y such that $x \leq y$.

Properties.

- If f is K -convex and a is a positive scalar, then af is k -convex for all $k \geq aK$.
- A convex function f is also 0-convex, and hence also K -convex for all $K \geq 0$.
- The sum of a K -convex function and a k -convex function is $(K + k)$ -convex.
- If v is K -convex, D is a random variable, then $G(y) := Ev(y - D)$ is K -convex.
- If f is K -convex, $x < y$, and $f(x) = K + f(y)$, then $f(z) \leq K + f(y)$ for all $z \in [x, y]$.

2.2 The proof of Preservation and Attainment

Now consider the function G_{N-1} :

$$G_{N-1}(y) = cy + H(y).$$

Clearly, G_{N-1} is convex and hence K -convex. We have

$$J_{N-1}(x) = \min[G_{N-1}(x), \min_{y > x} [K + G_{N-1}(y)]] - cx,$$

and it can be seen that

$$J_{N-1}(x) = \begin{cases} K + G_{N-1}(S_{N-1}) - cx, & \text{for } x < s_{N-1} \\ G_{N-1}(x) - cx, & \text{for } x \geq s_{N-1}, \end{cases} \quad (6)$$

where S_{N-1} minimizes $G_{N-1}(y)$ and s_{N-1} is the smallest value of y for which $G_{N-1}(y) = K + G_{N-1}(S_{N-1})$. Note that since $K > 0$, we have $s_{N-1} \neq S_{N-1}$ and furthermore the derivative of G_{N-1} at s_{N-1} is negative. As a result the left derivative of J_{N-1} at s_{N-1} is greater than the right derivative, and J_{N-1} is not convex. However, we will show that J_{N-1} is K -convex based on the fact that G_{N-1} is K -convex. To this end we must verify that for all $z \geq 0$, $b > 0$, and y , we have

$$K + J_{N-1}(y + z) \geq J_{N-1}(y) + z \left(\frac{J_{N-1}(y) - J_{N-1}(y - b)}{b} \right). \quad (7)$$