SEEM3470: Dynamic Optimization and Applications **2013–14 Second Term**

Tutorial 2: Introduction to Dynamic Programming

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Example 1. Let us define the function

$$f_N(a) = \max_R \left[x_1 x_2 \cdots x_N\right]$$

where R is the region determined by the conditions

$$\begin{aligned} &a. \quad x_1 + x_2 + \cdots + x_N = a, \quad a > 0; \\ &b. \quad x_i \geq 0. \end{aligned}$$

(1) Reformulate the above problem as a Dynamic Programming Problem.

(2) Using the result of (1), establish the arithmetic–geometric mean inequality, i.e.,

$$\left(\frac{x_1 + x_2 + \cdots + x_N}{N}\right)^N \geq x_1 x_2 \cdots x_N,$$

for $x_i \geq 0$, with equality only if $x_1 = x_2 = \cdots = x_N$.

Answers:

(1) Let us first consider the case $N = 1$, and it is easy to see that $f_1(a) = a$. Now, we move one step, say $N = 2$. By the description of the problem, we have

$$f_2(a) = \max_{\substack{x_1 + x_2 = a, \\ x_1 \geq 0, x_2 \geq 0}} \left[x_1 x_2\right].$$

Now let us choose an arbitrary $\bar{x}_2$, satisfied $0 \leq \bar{x}_2 \leq a$, then we have

$$x_1 \bar{x}_2 = \bar{x}_2(a - \bar{x}_2) = \bar{x}_2 f_1(a - \bar{x}_2).$$

Since $\bar{x}_2$ is arbitrary, then for any feasible x_2 the above relationship holds. As result, we have following equivalence,

$$f_2(a) = \max_{\substack{x_1 + x_2 = a, \\ x_1 \geq 0, x_2 \geq 0}} \left[x_1 x_2\right] = \max_{0 \leq x \leq a} x f_1(a - x).$$

We see that, when $N = 2$, the original problem can be reformulated as a dynamic programming problem. Now we assume that

$$f_N(a) = \max_{0 \leq x \leq a} x f_{N-1}(a - x), \; N \geq 2.$$

By definition, we have

$$f_N(a) = \max_{\substack{x_1 + x_2 + \cdots + x_N = a, \\ x_i \geq 0, \; i = 1, 2, \ldots, N}} \left[x_1 x_2 \cdots x_N\right].$$

Let $x_1^*, x_2^*, \ldots, x_N^*$ denote the optimal solutions of $f_N(a)$, and $\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_N$ denote the optimal solutions of the dynamic programming we assumed. The following two inequalities are easily verified by the optimality,

$$\max_{0\le x\le a} x f_{N-1}(a-x) = \hat{x}_1\hat{x}_2\cdots\hat{x}_N \le \max_{\substack{x_1+x_2+\cdots+x_N=a,\\ x_i\ge 0,\ i=1,2,\ldots,N}} [x_1x_2\cdots x_N] = x_1^*x_2^*\cdots x_N^*,$$

$$\max_{\substack{x_1+x_2+\cdots+x_N=a,\\ x_i\ge 0,\ i=1,2,\ldots,N}} [x_1x_2\cdots x_N] = x_1^*x_2^*\cdots x_N^* \le x_N^* f_{N-1}(a-x_N^*) \le \max_{0\le x\le a} x f_{N-1}(a-x) = \hat{x}_1\hat{x}_2\cdots\hat{x}_N.$$

Thus this two problems are equivalent. In summary, we have the original problem be formulated as the following dynamic programming problem

$$f_N(a) = \max_{0\le x\le a} x f_{N-1}(a-x),\ N \le 2,$$

with $f_1(a) = a$.

(2) Let us exploit more structures of the dynamic reformulation. Now $f_2(a) = \max_{0\le x\le a} x f_1(a-x) = \max_{0\le x\le a} x(a-x)$, which is to maximize an univariate quadratic function over the interval $[0, a]$. And it is easy to see the optimal $f_2(a) = \frac{a^2}{4}$, when $x_1 = x_2 = \frac{a}{2}$. We guess that

$$f_N(a) = \frac{a^N}{N^N},\ \text{with } x_1 = x_2 = \cdots = x_N.$$

We prove the above guess by mathematical induction. Since we already proved the case $N = 2$, let us suppose the following holds

$$f_{N-1}(a) = \frac{a^{N-1}|}{(N-1)^{N-1}},\ \text{with } x_1 = x_2 = \cdots = x_{N-1}.$$

By the result of (1), we have

$$f_N(a) = \max_{0\le x\le a} x f_{N-1}(a-x) = \max_{0\le x\le a} x\frac{(a-x)^{N-1}}{(N-1)^{N-1}}.$$

Now the problem becomes to maximize a univariate nonlinear function $f(x) = x\frac{(a-x)^{N-1}}{(N-1)^{N-1}}$ over the interval $[0, a]$. Let us compute the first order derivative $f'(x) = \frac{(a-x)^{N-1}}{(N-1)^{N-1}} - x\frac{(a-x)^{N-2}}{(N-1)^{N-2}}$, and set it to zero we have

$$(a-x)^{N-2}(x - \frac{a-x}{N-1}) = 0.$$

Then we have $x = a$ or $x = \frac{a}{N}$. Clearly we can not have $x = a$, otherwise the objective value will be 0. Then we have $x = \frac{a}{N}$, and by the inductive hypothesis we have the rest $x_i = \frac{a}{N}$, which results in

$$f_N(a) = \frac{a^N}{N^N},\ \text{with } x_1 = x_2 = \cdots = x_N.$$

By the above result, we have

$$x_1x_2\cdots x_N \le f_N(x_1+x_2+\cdots+x_N) = \left(\frac{x_1+x_2+\cdots+x_N}{N}\right)^N,$$

and the equality holds when $x_1 = x_2 = \cdots = x_N$.

Example 2 Let us define the function

$$f_N(a) = \min_R \sum_{i=1}^{N} x_i^p,$$

for some $p > 0$ and R is the region defined by

$$\begin{array}{ll} a. & \sum_{i=1}^{N} x_i \geq a,\ a > 0; \\ b. & x_i \geq 0,\ i = 1, 2, \ldots, N. \end{array}$$

Show that $f_N(a)$ satisfies the recurrence relation

$$f_N(a) = \min_{0 \leq x \leq a} \left[x^p + f_{N-1}(a - x) \right],\ N \geq 2,$$

with $f_1(a) = a^p$.