

SOLUTION FOR HOMEWORK 1, SEG5580

September 11, 2009

PROBLEM 1.(a):

$$\begin{aligned}P(F(X) \leq x) &= P(X \leq F^{-1}(x)) \\&= F(F^{-1}(x)) \\&= x\end{aligned}$$

PROBLEM 1.(b):

$$\begin{aligned}P(F^{-1}(U) \leq x) &= P(U \leq F(x)) \\&= F(x)\end{aligned}$$

PROBLEM 2.(a):

$$\begin{aligned}P(X_1 + X_2 = n) &= \sum_{k=0}^n P(X_1 = k)P(X_2 = n - k) \\&= \sum_{k=0}^n \frac{n! \lambda_1^k \lambda_2^{n-k}}{k!(n-k)!} \frac{e^{-(\lambda_1 + \lambda_2)}}{n!} \\&= (\lambda_1 + \lambda_2)^n \frac{e^{-(\lambda_1 + \lambda_2)}}{n!}\end{aligned}\tag{1}$$

Where the last equality holds because $\sum_{k=0}^n \frac{n! \lambda_1^k \lambda_2^{n-k}}{k!(n-k)!} = (\lambda_1 + \lambda_2)^n$

PROBLEM 2.(b):

$$\begin{aligned} P(X_1 = k, X_1 + X_2 = n) &= P(X_1 = k)P(X_2 = n - k) \\ &= \frac{\lambda_1^k e^{-\lambda_1}}{k!} \frac{\lambda_2^{n-k} e^{-\lambda_2}}{(n-k)!} \end{aligned} \quad (2)$$

Thus, combine (1) and (2) together, we get:

$$\begin{aligned} P(X_1 = k / X_1 + X_2 = n) &= \frac{P(X_1 = k, X_1 + X_2 = n)}{P(X_1 + X_2 = n)} \\ &= \frac{n!}{k!(n-k)!} \frac{\lambda_1^k \lambda_2^{n-k}}{(\lambda_1 + \lambda_2)^n} \end{aligned}$$

PROBLEM 3:

$$\begin{aligned} V(X) &= V(X - a/2) \\ &\leq E(X - a/2)^2 \\ &\leq a^2/4 \end{aligned}$$

where the last inequality holds because $P(-a/2 \leq X - a/2 \leq a/2) = 1$

PROBLEM 4.(a):

$$\begin{aligned} F_{i,n}(x) &= P(\text{at least } i \text{ items from } X_1, X_2, \dots, X_n \leq x) \\ &= P(\text{at least } i \text{ items from } X_1, X_2, \dots, X_n \leq x | X_n > x)P(X_n > x) \\ &\quad + P(\text{at least } i \text{ items from } X_1, X_2, \dots, X_n \leq x | X_n \leq x)P(X_n \leq x) \end{aligned}$$

Since $P(X_n > x) = (1 - F(x))$, $P(X_n \leq x) = F(x)$, and

$$\begin{aligned}
& P(\text{at least } i \text{ items from } X_1, X_2, \dots, X_n \leq x | X_n > x) \\
&= P(\text{at least } i \text{ items from } X_1, X_2, \dots, X_{n-1} \leq x | X_n > x) \\
&= P(\text{at least } i \text{ items from } X_1, X_2, \dots, X_{n-1} \leq x) \\
&= F_{i,n-1}(x)
\end{aligned}$$

$$\begin{aligned}
& P(\text{at least } i \text{ items from } X_1, X_2, \dots, X_n \leq x | X_n \leq x) \\
&= P(\text{at least } i-1 \text{ items from } X_1, X_2, \dots, X_{n-1} \leq x | X_n \leq x) \\
&= P(\text{at least } i-1 \text{ items from } X_1, X_2, \dots, X_{n-1} \leq x) \\
&= F_{i-1,n-1}(x)
\end{aligned}$$

We arrive at the desired result.

PROBLEM 4,(b):

$$\begin{aligned}
F_{i,n-1}(x) &= P(\text{at least } i \text{ items from } X_1, X_2, \dots, X_{n-1} \leq x) \\
&= P(X_n \text{ among the } i \text{ smallest of } X_1, X_2, \dots, X_n) \\
&\cdot P(\text{at least } i \text{ items from } X_1, X_2, \dots, X_{n-1} \leq x | X_n \text{ among the } i \text{ smallest of } X_1, X_2, \dots, X_n) \\
&+ P(X_n \text{ not among the } i \text{ smallest of } X_1, X_2, \dots, X_n) \\
&\cdot P(\text{at least } i \text{ items from } X_1, X_2, \dots, X_{n-1} \leq x | X_n \text{ not among the } i \text{ smallest of } X_1, X_2, \dots, X_n)
\end{aligned}$$

Since $X_1, X_2, \dots, X_n$ is i.i.d., $P(X_n \text{ among the } i \text{ smallest of } X_1, X_2, \dots, X_n) = \frac{i}{n}$,
and $P(X_n \text{ not among the } i \text{ smallest of } X_1, X_2, \dots, X_n) = \frac{n-i}{n}$.

$$\begin{aligned}
& P(\text{at least } i \text{ items from } X_1, X_2, \dots, X_{n-1} \leq x | X_n \text{ among the } i \text{ smallest of } X_1, X_2, \dots, X_n) \\
&= P(\text{at least } i+1 \text{ items from } X_1, X_2, \dots, X_n \leq x | X_n \text{ among the } i \text{ smallest of } X_1, X_2, \dots, X_n) \\
&= P(\text{at least } i+1 \text{ items from } X_1, X_2, \dots, X_n \leq x) \\
&= F_{i+1,n}(x)
\end{aligned}$$

$$\begin{aligned}
& P(\text{at least } i \text{ items from } X_1, X_2, \dots, X_{n-1} \leq x | X_n \text{ not among the } i \text{ smallest of } X_1, X_2, \dots, X_n) \\
&= P(\text{at least } i \text{ items from } X_1, X_2, \dots, X_n \leq x | X_n \text{ not among the } i \text{ smallest of } X_1, X_2, \dots, X_n) \\
&= P(\text{at least } i \text{ items from } X_1, X_2, \dots, X_n \leq x) \\
&= F_{i,n}(x)
\end{aligned}$$

Then, we have $F_{i,n-1}(x) = \frac{i}{n} F_{i+1,n}(x) + \frac{n-i}{n} F_{i,n}(x)$

PROBLEM 5,(a):

Let X_n be the number of white balls after the operation has been repeated n times.

$$\begin{aligned}
E(X_{n+1}|X_n) &= \frac{X_n}{a+b} X_n + \frac{a+b-X_n}{a+b} (X_n+1) \\
&= X_n \left(1 - \frac{1}{a+b}\right) + 1
\end{aligned}$$

Taking expectation, we get $M_{n+1} = M_n \left(1 - \frac{1}{a+b}\right) + 1$.

PROBLEM 5,(b):

Observe $[M_n - (a+b)] = [1 - \frac{1}{a+b}]^n [a - (a+b)]$, it's simple to see $M_n = (a+b) - b[1 - \frac{1}{a+b}]^n$

PROBLEM 5,(c):

Since $P(\text{the } (n+1)\text{st ball drawn is white} | X_n) = \frac{X_n}{a+b}$, taking expectation, we get the result: $P(\text{the } (n+1)\text{st ball drawn is white}) = \frac{M_n}{a+b}$.

PROBLEM 6:

By induction, we assume $P(N = n) = \frac{e^{-\lambda} \lambda^n}{n!}$ is true for n , and pass to $n+1$:

$$\begin{aligned}
N = n+1 &\Leftrightarrow U_1 U_2 \dots U_{n+1} > e^{-\lambda} > U_1 U_2 \dots U_{n+1} U_{n+2} \\
&\Leftrightarrow U_2 \dots U_{n+1} > \frac{e^{-\lambda}}{U_1} > U_2 \dots U_{n+1} U_{n+2}
\end{aligned} \tag{3}$$

when $U_1 < e^{-\lambda}$, $\frac{e^{-\lambda}}{U_1} > 1$, and $U_2 \dots U_{n+1} > 1$ is impossible, thus $P(N = n + 1 | U_1 < e^{-\lambda}) = 0$.

when $U_1 \geq e^{-\lambda}$, $\frac{e^{-\lambda}}{U_1} = e^{-(\lambda + \ln U_1)} \leq 1$

$$\begin{aligned} P(N = n + 1) &= P(N = n + 1 | U_1 \geq e^{-\lambda}) \\ &= \int_{e^{-\lambda}}^1 \frac{e^{-(\lambda + \ln U_1)}}{n!} (\lambda + \ln U_1)^n dU_1 \\ &= \int_{e^{-\lambda}}^1 \frac{e^{-\lambda}}{n!} (\lambda + \ln U_1)^n dU_1 \frac{1}{U_1} \end{aligned}$$

By changing variable $\lambda + \ln U_1 = y$, $P(N = n + 1) = \int_0^\lambda \frac{e^{-y}}{n!} y^n dy = \frac{e^{-\lambda} \lambda^{n+1}}{(n+1)!}$

PROBLEM 7.(a):

$$\begin{aligned} \alpha &= P(\text{ever reaches 1} / \text{starting from 0}) \\ &= P(\text{ever reaches 1} / \text{starting from 1}) P(\text{move to 1} / \text{starting from 0}) \\ &\quad + P(\text{ever reaches 1} / \text{starting from -1}) P(\text{move to -1} / \text{starting from 0}) \\ &= 1 \cdot p + P(\text{ever reaches 1} / \text{starting from -1}) \cdot (1 - p) \\ &= 1 \cdot p + P(\text{ever reaches 1} / \text{starting from 0}) P(\text{ever reaches 0} / \text{starting from -1}) \cdot (1 - p) \\ &= p + (1 - p) \alpha^2 \end{aligned}$$

Where the last equality holds since the case from -1 to 0 is the same as the case from 0 to 1.

PROBLEM 7.(b):

$\alpha = 1$ and $\alpha = p/(1 - p)$ are two solutions for the above equation. For the case $p > 1/2$, we must choose $\alpha = 1$ since $p/(1 - p) > 1/2$; For the case $p \leq 1/2$, we must choose $\alpha = p/(1 - p)$ since we should have $\lim_{p \rightarrow 0} \alpha = 0$.

PROBLEM 7.(c):

$$\begin{aligned}
& P(\text{ever reaches } n/\text{starting from } 0) \\
&= P(\text{ever reaches } n/\text{starting from } 1)P(\text{ever reaches } 1/\text{starting from } 0) \\
&= \alpha P(\text{ever reaches } n/\text{starting from } 1) \\
&= \alpha P(\text{ever reaches } n/\text{starting from } 2)P(\text{ever reaches } 2/\text{starting from } 1) \\
&= \alpha^2 P(\text{ever reaches } n/\text{starting from } 2) \\
&= \dots\dots \\
&= \alpha^{n-1} P(\text{ever reaches } n/\text{starting from } n-1) \\
&= \alpha^n
\end{aligned}$$

PROBLEM 7.(d):

By Bayes' rule,

$$\begin{aligned}
P(\text{move to } i+1|\text{ever reaches } n) &= \frac{P(\text{move to } i+1, \text{ever reaches } n)}{P(\text{ever reaches } n)} \\
&= \frac{P(\text{ever reaches } n|\text{move to } i+1)P(\text{move to } i+1)}{P(\text{ever reaches } n)}
\end{aligned}$$

Since the current place is i , $P(\text{move to } i+1) = p$, $P(\text{ever reaches } n) = \alpha^{n-i}$, and $P(\text{ever reaches } n|\text{move to } i+1) = \alpha^{n-i-1}$, together with $\alpha = p/(1-p)$, we get the result $P(\text{move to } i+1|\text{ever reaches } n) = 1-p$.

PROBLEM 8.(a):

$$\begin{aligned}
\int f_t(x)dx &= \int \frac{e^{tx}f(x)}{M(t)}dx \\
&= \frac{M(t)}{M(t)} \\
&= 1
\end{aligned}$$

and $f_t(x) \geq 0$. Thus f_t is a probability density function.

PROBLEM 8.(b):

$$\begin{aligned}M(t)E[e^{-tX_t}h(X_t)] &= M(t) \int e^{-tx}h(x) \frac{e^{tx}f(x)}{M(t)}dx \\&= \int h(x)f(x)dx \\&= E[h(X)]\end{aligned}$$

PROBLEM 8.(c):

$$\begin{aligned}M(t)e^{-ta}P(X_t > a) &= M(t)e^{-ta} \int_{x>a} \frac{e^{tx}f(x)}{M(t)}dx \\&= \int_{x>a} e^{t(x-a)}f(x)dx \\&> \int_{x>a} f(x)dx \\&= P(X > a)\end{aligned}$$

Where the inequality holds because when $x > a$, $e^{t(x-a)} > 1$.

PROBLEM 8.(d):

$$\begin{aligned}\frac{d}{dt}M(t)e^{-ta} &= M'(t)e^{-ta} - aM(t)e^{-ta} \\&= e^{-ta}[M'(t) - aM(t)]\end{aligned}$$

we shall show $\frac{d}{dt}M(t)e^{-ta} \stackrel{t=t^*}{=} 0$; and $\frac{d^2}{dt^2}M(t)e^{-ta} \geq 0$.

$$\begin{aligned}
E(X_{t^*}) &= a \\
\Rightarrow \int \frac{x e^{t^* x} f(x)}{M(t^*)} dx &= a \\
\Rightarrow \int e^{t^* x} x f(x) dx &= a M(t^*) \\
\Rightarrow E[X e^{t^* X}] &= a M(t^*) \\
\Rightarrow M'(t^*) &= a M(t^*) \\
\Rightarrow \frac{d}{dt} M(t) e^{-ta} \Big|_{t=t^*} &= 0
\end{aligned}$$

since $M'(t) = E[X e^{tX}]$, and $M''(t) = E[X^2 e^{tX}]$, we have:

$$\begin{aligned}
\frac{d^2}{dt^2} M(t) e^{-ta} &= M''(t) e^{-ta} - 2a M'(t) e^{-ta} + a^2 M(t) e^{-ta} \\
&= e^{-ta} [M''(t) - 2a M'(t) + a^2 M(t)] \\
&= e^{-ta} (E[X^2 e^{tX}] - 2a E[X e^{tX}] + a^2 E[e^{tX}]) \\
&= e^{-ta} E[(X^2 - 2aX + a^2) e^{tX}] \\
&= e^{-ta} E[(X - a)^2 e^{tX}] \\
&\geq 0
\end{aligned}$$

then we get the conclusion: $\min_t M(t) e^{-ta} = M(t^*) e^{-t^* a}$