

SOLUTION FOR HOMEWORK 2, SEG5580

September 25, 2009

PROBLEM 1.:

$$\begin{aligned}E[N(t) \cdot N(t+s)] &= EE[N(t) \cdot (N(t+s) - N(t) + N(t)) | N(t)] \\&= E[N(t) \cdot \lambda s + N(t)^2] \\&= \lambda^2 ts + \lambda^2 t^2 + \lambda t\end{aligned}$$

PROBLEM 2.:

$N_1(0) + N_2(0) = 0$ is obvious.

Since $N_1(t+s) - N_1(t)$ is independent with $N_1(t)$; $N_2(t+s) - N_2(t)$ is independent with $N_2(t)$, then $N_1(t+s) - N_1(t) + N_2(t+s) - N_2(t)$ is independent with $N_1(t) + N_2(t)$. Thus the process $N_1(t) + N_2(t)$ has independent increments.

Since $N_1(t+s) - N_1(t) \sim \text{Poisson}(\lambda_1 s)$, $N_2(t+s) - N_2(t) \sim \text{Poisson}(\lambda_2 s)$, we have $N_1(t+s) + N_2(t+s) - (N_1(t) + N_2(t)) \sim \text{Poisson}((\lambda_1 + \lambda_2)s)$. This is what we have proved last homework.

PROBLEM 3:

Assume the person arrives at an arbitrary time t , and denote S_n as the n th arrival time. Since each inter-arrival time is from $\text{Exp}(\lambda)$, we have the distribution for residual life $Y \triangleq S_{N_t+1} - t$:

$$\begin{aligned}
P(Y > y) &= P(S_{N_t+1} - S_{N_t} > t - S_{N_t} + y | S_{N_t+1} - S_{N_t} > t - S_{N_t}) \\
&= \int_u P(S_{N_t+1} - S_{N_t} > t - u + y | S_{N_t+1} - S_{N_t} > t - u) P(S_{N_t} \in du) \\
&= \int_u P(S_{N_t+1} - S_{N_t} > y) P(S_{N_t} \in du) \\
&= P(S_{N_t+1} - S_{N_t} > y)
\end{aligned}$$

Where the 3rd equality holds because of the memoryless property of exponential distribution. Thus we have proved the residual life $Y \sim \text{Exp}(\lambda)$. Obviously the waiting time satisfies the equation:

$$\text{waiting time} = T1_{(Y>T)} + (Y + \text{new waiting time})1_{(Y<T)}$$

Since 'waiting time' and 'new waiting time' has the same distribution, and 'new waiting time' is independent with Y , taking expectation on both sides, we get:

$$\begin{aligned}
EW &= TP(Y > T) + EY1_{(Y<T)} + EWP(Y < T) \\
\Rightarrow EW &= \frac{TP(Y > T) + EY1_{(Y<T)}}{P(Y > T)}
\end{aligned}$$

And $P(Y > T) = e^{-\lambda T}$, $EY1_{(Y<T)} = \int_0^T y\lambda e^{-\lambda y} dy = \frac{-\lambda T e^{-\lambda T} + 1 - e^{-\lambda T}}{\lambda}$. Put all these into the above equation, we get $EW = \frac{1 - e^{-\lambda T}}{\lambda e^{-\lambda T}}$.

PROBLEM 4.(a)

As in the Problem 4, we have proved the residual life $Y \sim \text{Exp}(\lambda)$.

Now the waiting time is $(s + W)1_{(Y>s)} + (Y + R)1_{(Y\leq s)}$, and the expected waiting time is $\int_s^\infty \lambda e^{-\lambda x} dx (s + W) + \int_0^s (x + R)\lambda e^{-\lambda x} dx$. After simple calculation, it is $e^{-\lambda s}(W - R - \frac{1}{\lambda}) + R + \frac{1}{\lambda}$

PROBLEM 4.(b)

The conclusion is obvious from the expression $e^{-\lambda s}(W - R - \frac{1}{\lambda}) + R + \frac{1}{\lambda}$

PROBLEM 5,(a):

$$\begin{aligned}
& f_{X_{(i)}}(x) \\
&= P(X_{(i)} \in dx)/dx \\
&= P(i-1 \text{ items from } X_1, \dots, X_n < x, \text{ one item from } X_1, \dots, X_n \in dx, \\
&\quad n-i \text{ items from } X_1, \dots, X_n > x)/dx \\
&= \frac{n!}{(i-1)!(n-i)!} (F(x))^{i-1} (1-F(x))^{n-i} f(x)
\end{aligned}$$

PROBLEM 5,(b):

$X_{(i)}$ less than x if and only if at least i items are less than x .

PROBLEM 5,(c):

$$P(X_{(i)} \leq x) = \sum_{j=i}^n C_n^j (F(x))^j (1-F(x))^{n-j}.$$

PROBLEM 5,(d):

Given $N_t = n$, S_i is distributed as the i th smallest of n uniform $[0,t]$ random variables. From the result of (a), S_i has density function: $f_S(x) = \frac{n!}{(i-1)!(n-i)!} \left(\frac{x}{t}\right)^{i-1} \left(1 - \frac{x}{t}\right)^{n-i} \frac{1}{t}$.

$$\begin{aligned}
& E[S_i | N_t = n] \\
&= \int_0^t x f_S(x) dx \\
&= \frac{n!}{(i-1)!(n-i)!} t \int_0^t \left(\frac{x}{t}\right)^{i-1} \left(1 - \frac{x}{t}\right)^{n-i} d\frac{x}{t} \\
&= \frac{n!}{(i-1)!(n-i)!} t \int_0^1 u^{i-1} (1-u)^{n-i} du \\
&= \frac{i}{n+1} t
\end{aligned}$$

The last step is because $\int_0^1 u^i (1-u)^{n-i} du = \text{beta}(i+1, n-i+1) = \frac{i!(n-i)!}{(n+1)!}$.

PROBLEM 6:

Let's divide the cars into two types:

Type One: Located in the interval (a,b) at time t

Type Two: Not located in the interval (a,b) at time t

If a car enters the highway at time $s(s < t)$, it will be classified as type one with probability $p(s) = P(a < V(t-s) < b) = F(\frac{b}{t-s}) - F(\frac{a}{t-s})$. Then the number of cars of type one at time t is Poisson random variable with mean λtp , where $p = \frac{1}{t} \int_0^t p(s) ds = \frac{1}{t} \int_0^t [F(\frac{b}{t-s}) - F(\frac{a}{t-s})] ds$.