

SOLUTION FOR HOMEWORK 3, SEG5580

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PROBLEM 1.:

Let τ be the first time the chain hits j ,

$$\begin{aligned} p_{ij}^n &= P(X_{n+m} = j | X_m = i) \\ &= \sum_{k=0}^n P(X_{n+m} = j | X_m = i, \tau = m+k) P(\tau = m+k | X_m = i) \\ &= \sum_{k=0}^n P(X_{n+m} = j | X_{m+k} = j) P(\tau = m+k | X_m = i) \\ &= \sum_{k=0}^n f_{ij}^k p_{jj}^{n-k} \end{aligned}$$

PROBLEM 2.(a):

$p_{ij/k}^n$ is the n -step transition probability, from i to j , such that the path never hit k .

PROBLEM 2.(b):

Let τ be the last time the chain hits i ,

$$\begin{aligned}
& p_{ij}^n \\
&= P(X_{n+m} = j | X_m = i) \\
&= \sum_{k=0}^n P(X_{n+m} = j | X_m = i, \tau = m+k) P(\tau = m+k | X_m = i) \\
&= \sum_{k=0}^n P(X_{n+m} = j, X_l \neq i, l = m+k+1, m+k+2, \dots, m+n-1 | X_{m+k} = i) \\
&\quad \cdot P(\tau = m+k | X_m = i) \\
&= \sum_{k=0}^n f_{ij}^k p_{jj}^{n-k}
\end{aligned}$$

PROBLEM 3:

Assume the chain starts from 0. We calculate the probability p_{00}^m . When m is odd number, the probability is zero; When m is even number, assume U, D, F, B, L, R be the number of steps that the chain choose upward, downward, forward, backward, left, right.

$$\begin{aligned}
& p_{00}^{2n} \\
&= P(U = D, F = B, L = R) \\
&= 6^{-2n} \sum_{j,k} \frac{(2n)!}{(j!k!(n-j-k)!)^2} \\
&= 2^{-2n} C_n^{2n} \sum_{j,k} \left(3^{-n} \frac{n!}{j!k!(n-j-k)!} \right)^2 \\
&\leq 2^{-2n} C_n^{2n} \max_{j,k} 3^{-n} \frac{n!}{j!k!(n-j-k)!}
\end{aligned}$$

Where the last inequality is due to $3^{-n} \frac{n!}{j!k!(n-j-k)!} \geq 0$ and $\sum_{j,k} 3^{-n} \frac{n!}{j!k!(n-j-k)!} = 1$.

The next step is to show $\max_{j,k} 3^{-n} \frac{n!}{j!k!(n-j-k)!} = O(\frac{1}{n})$ as $n \rightarrow \infty$. We first note that $3^{-n} \frac{n!}{j!k!(n-j-k)!}$ is maximized when $j, k, n-j-k$ is as close as possible

to $n/3$; then apply Stirling approximation, we get:

$$\begin{aligned}
& 3^{-n} \frac{n!}{j!k!(n-j-k)!} \\
& \stackrel{j=k \simeq n/3}{\simeq} 3^{-n} \frac{n!}{\frac{n}{3}! \frac{n}{3}! \frac{n}{3}!} \\
& \simeq 3^{-n} \frac{n^n}{\frac{n}{3} \frac{n}{3} \frac{n}{3} \frac{n}{3} \frac{n}{3}} \sqrt{\frac{n}{\frac{n}{3} \frac{n}{3} \frac{n}{3}}} \frac{1}{2\pi} \\
& = O\left(\frac{1}{n}\right)
\end{aligned}$$

The last step is to show $2^{-2n}C_n^{2n} = O\left(\frac{1}{\sqrt{n}}\right)$ as $n \rightarrow \infty$. Simply apply Stirling approximation,

$$\begin{aligned}
& 2^{-2n}C_n^{2n} \\
& = 2^{-2n} \frac{(2n)!}{n!n!} \\
& \simeq 2^{-2n} \frac{(2n)^{2n}}{n^n n^n} \sqrt{\frac{2n}{nn}} \sqrt{2\pi} \\
& = O\left(\frac{1}{\sqrt{n}}\right)
\end{aligned}$$

Now we see $p_{00}^{2n} \leq O\left(\frac{1}{n^{1.5}}\right)$, and $\sum_{n=1}^{\infty} p_{00}^{2n} < \infty$. We have proved 0 is transient! And since any point $x \leftrightarrow 0$, any point x is also transient.

PROBLEM 4.(a)

$\{R_n, n \geq 1\}$ is a Markov chain, a new record occurs if and only if a new sample X_m greater than the current record. The record history doesn't matter. Assume the current record, say, R_n is i , the next record can only take value in $\{i+1, i+2, \dots\}$. The transition density is

$$\begin{aligned}
& p_{ij} \\
& = P(R_{n+1} = j | R_n = i) \\
& = \frac{a_j}{\sum_{k=i+1}^{\infty} a_k}
\end{aligned}$$

PROBLEM 4.(b)

$\{T_n, n \geq 1\}$ is not a Markov chain, but $\{(R_n, T_n), n \geq 1\}$ is. Assume $R_{n-1} = i, T_{n-1} = l$. The transition density of R_n is known from (a); The distribution of T_n , conditioned on $R_n = j$, is Geometry distribution. Since each new sample after R_n has probability $\sum_{k=j+1}^{\infty} a_k$ to be a new record, and has probability $\sum_{k=0}^j a_k$ not to be a new record. thus the conditioned density $P(T_n = m | R_n = i) = (\sum_{k=0}^j a_k)^{m-1} (\sum_{k=j+1}^{\infty} a_k)$. Then the transition density of $\{(R_n, T_n)\}$ is:

$$\begin{aligned} & P(R_n = j, T_n = m | R_{n-1} = i, T_{n-1} = l) \\ &= P(R_n = j | R_{n-1} = i, T_{n-1} = l) P(T_n = m | R_n = j) \\ &= \frac{a_j}{\sum_{k=i+1}^{\infty} a_k} \left(\sum_{k=0}^j a_k \right)^{m-1} \left(\sum_{k=j+1}^{\infty} a_k \right) \end{aligned}$$

PROBLEM 4.(c)

Since $S_n = S_{n-1} + T_n$, S_n depends on the sequence $S_1, S_2, \dots, S_{n-1}$ only through S_{n-1} . It is Markov chain. Denote the distribution function of X as $F(\cdot)$. Assume $S_{n-1} = l$, which means the current record R_n is the maximum out of $\{X_1, X_2, \dots, X_l\}$. It indicates the distribution function of R_n is $P(R_n \leq x) = (F(x))^l$. Conditioned on $R_n = x$, the distribution of T_n is Geometry, since each new sample has the probability $P(X > R_n)$ to be a new record.

$$\begin{aligned} & P(X > R_n) \\ &= EP(X > R_n | R_n) \\ &= \int (1 - F(x)) d(F(x))^l \\ &= \frac{1}{l+1} \end{aligned}$$

Then the transition density of S_n is:

$$\begin{aligned} & P(S_n = l + m | S_{n-1} = l) \\ &= P(T_n = m | S_{n-1} = l) \\ &= \left(1 - \frac{1}{l+1} \right)^{m-1} \left(\frac{1}{l+1} \right) \end{aligned}$$

PROBLEM 5:

$$\begin{aligned}
& P(X_k = i_k | X_j = i_j, \text{ for all } j \neq k) \\
&= \frac{P(X_j = i_j, j \leq k-1, X_k = i_k, X_j = i_j, j \geq k+1)}{P(X_j = i_j, j \leq k-1, X_j = i_j, j \geq k+1)} \\
&= \frac{P(X_k = i_k | X_j = i_j, j \leq k-1) P(X_j = i_j, j \geq k+1 | X_j = i_j, j \leq k-1, X_k = i_k)}{P(X_j = i_j, j \geq k+1 | X_j = i_j, j \leq k-1)} \\
&= \frac{P(X_k = i_k | X_{k-1} = i_{k-1}) P(X_j = i_j, j \geq k+1 | X_k = i_k)}{P(X_j = i_j, j \geq k+1 | X_{k-1} = i_{k-1})} \\
&= \frac{P(X_k = i_k | X_{k-1} = i_{k-1}) P(X_j = i_j, j \geq k+1 | X_k = i_k)}{\sum_{i_k} P(X_k = i_k | X_{k-1} = i_{k-1}) P(X_j = i_j, j \geq k+1 | X_k = i_k)} \\
&= \frac{P(X_k = i_k | X_{k-1} = i_{k-1}) P(X_{k+1} = i_{k+1} | X_k = i_k) P(X_j = i_j, j \geq k+2 | X_{k+1} = i_{k+1})}{\sum_{i_k} P(X_k = i_k | X_{k-1} = i_{k-1}) P(X_{k+1} = i_{k+1} | X_k = i_k) P(X_j = i_j, j \geq k+2 | X_{k+1} = i_{k+1})} \\
&= \frac{P(X_k = i_k | X_{k-1} = i_{k-1}) P(X_{k+1} = i_{k+1} | X_k = i_k)}{\sum_{i_k} P(X_k = i_k | X_{k-1} = i_{k-1}) P(X_{k+1} = i_{k+1} | X_k = i_k)} \\
&= \frac{P(X_{k-1} = i_{k-1}, X_k = i_k, X_{k+1})}{\sum_{i_k} P(X_{k-1} = i_{k-1}, X_k = i_k, X_{k+1})} \\
&= \frac{P(X_{k-1} = i_{k-1}, X_k = i_k, X_{k+1})}{P(X_{k-1} = i_{k-1}, X_{k+1})} \\
&= P(X_k = i_k | X_{k-1} = i_{k-1}, X_{k+1})
\end{aligned}$$

PROBLEM 6:

$$\begin{aligned}
& P(\text{she wins the next gamble} | \text{present fortune is } i, \text{ she eventually reaches } N) \\
&= \frac{P(\text{she wins the next gamble, present fortune is } i, \text{ she eventually reaches } N)}{P(\text{present fortune is } i, \text{ she eventually reaches } N)} \\
&= \frac{P(\text{she wins the next gamble}) P(\text{present fortune is } i+1, \text{ she eventually reaches } N)}{P(\text{present fortune is } i, \text{ she eventually reaches } N)} \\
&= \frac{p \cdot f_{i+1}}{f_i}
\end{aligned}$$

From textbook, $f_i = \frac{1-(q/p)^i}{1-(q/p)^N}$ when $p \neq 1/2$; and $f_i = \frac{i}{N}$ when $p = 1/2$. Plug this result into the above equation we get the answer.