

SOLUTION FOR HOMEWORK 4, SEG5580

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PROBLEM 1(a).

It's enough to prove (b)

PROBLEM 1(b).

$$\begin{aligned}
 & \sum_{n=1}^{\infty} p_{ij}^n \\
 &= E[\text{number of arrives at } j | X_0 = i] \\
 &= (1 + E[\text{number of arrives at } j | X_0 = j]) \cdot P(j \text{ is attainable} | X_0 = i) \\
 &= (1 + \sum_{n=1}^{\infty} p_{jj}^n) f_{ij}
 \end{aligned}$$

PROBLEM 2.

In the case $p \neq 1/2$

from $E[\sum_{i=1}^{\tau} X_i] = E[\tau]E[X_i]$, and $E[X_i] = p(1) + q(-1) = 2p - 1$, we get $E[\tau] = \frac{E[\sum_{i=1}^{\tau} X_i]}{2p-1}$. $\sum_{i=1}^{\tau} X_i$ takes value $N - i$ with probability f_{iN} , takes value $-i$ with probability $f_{i0} = 1 - f_{iN}$, and $f_{iN} = \frac{1-(q/p)^i}{1-(q/p)^N}$. We get $E[\tau] = \frac{Nf_{iN}-i}{2p-1} = \frac{N \frac{1-(q/p)^i}{1-(q/p)^N} - i}{2p-1}$

In the case $p = 1/2$

Let $t_i = E[\tau|X_0 = i]$, then $t_i = 1/2(t_{i-1} + 1) + 1/2(t_{i+1} + 1)$, for $1 \leq i \leq N-1$, $t_0 = t_N = 0$. Solve this linear system, we get $t_i = i(N-i)$.

PROBLEM 3.(a):

$\mu_{i,n} = (1 + \mu_{i+1,n})p + (1 + \mu_{i-1,n})q$ for $1 \leq i \leq n-1$; $\mu_{0,n} = \mu_{1,n} + 1, \mu_{n,n} = \mu_{n-1,n} + 1$.

PROBLEM 3.(b):

$m_i = 1 \cdot p + (1 + m_{i-1} + m_i) \cdot q$, for $1 \leq i \leq n-1$, $m_0 = 1$.

$$\begin{aligned} m_i &= 1 \cdot p + (1 + m_{i-1} + m_i) \cdot q \\ \Rightarrow m_i &= 1/p + (1-p)/pm_{i-1} \\ \Rightarrow m_i &= \frac{1}{2p-1} [1 - (q/p)^i] + (q/p)^i \end{aligned} \quad (1)$$

PROBLEM 3.(c):

$\mu_{i,n} = m_i + m_{i+1} + \dots + m_{n-1}$.

PROBLEM 3.(d):

$E[X_j] = 1 + E[\tau|X_0 = 1]$, and we have calculated $E[\tau|X_0 = 1]$ in Problem 2. Thus

$$E[X_j] = \begin{cases} \frac{n^{\frac{1-(q/p)}{1-(q/p)^n}} - 1}{2p-1} + 1, & p \neq 1/2 \\ n, & p = 1/2 \end{cases}$$

PROBLEM 3.(e):

When we stand at state 1, we shall find N follows Geometry distribution:
 $P(N = k) = (f_{10})^{k-1} f_{1n}$

$$f_{1n} = \begin{cases} \frac{1-(q/p)}{1-(q/p)^n}, & p \neq 1/2 \\ 1/n, & p = 1/2 \end{cases}$$

and $f_{10} = 1 - f_{1n}$. Thus

$$E[N] = \frac{1}{f_{1n}} = \begin{cases} \frac{1-(q/p)^n}{1-(q/p)}, & p \neq 1/2 \\ n, & p = 1/2 \end{cases}$$

PROBLEM 3.(f):

$$\begin{aligned} \mu_{0,n} &= m_0 + m_1 + \dots m_{n-1} \\ &= \frac{n}{2p-1} + \frac{2p(p-1)[1-(q/p)^n]}{(2p-1)^2} \end{aligned}$$

$$\begin{aligned} \mu_{i,n} &= m_i + m_{i+1} + \dots m_{n-1} \\ &= \frac{n-i}{2p-1} + \frac{2p(p-1)[(q/p)^i - (q/p)^n]}{(2p-1)^2} \end{aligned}$$

PROBLEM 4.(a):

State A: (1,2); State B: (2,1); State C: (1,1) or (2,2)(end of hunt); the transition matrix:

$$M = \begin{pmatrix} .28 & .18 & .54 \\ .18 & .28 & .54 \\ 0 & 0 & 1 \end{pmatrix}$$

PROBLEM 4.(b):

Decompose M as

$$\begin{aligned}
 M &= U \begin{pmatrix} .46 & 0 & 0 \\ 0 & .1 & 0 \\ 0 & 0 & 1 \end{pmatrix} U^{-1} \\
 U &= \begin{pmatrix} .7071 & -.7071 & .5754 \\ .7071 & .7071 & .5754 \\ 0 & 0 & .5754 \end{pmatrix} \\
 U^{-1} &= \begin{pmatrix} .7071 & .7071 & -1.4142 \\ -.7071 & .7071 & 0 \\ 0 & 0 & 1.7321 \end{pmatrix} \\
 \Rightarrow M^n &= U \begin{pmatrix} .46^n & 0 & 0 \\ 0 & .1^n & 0 \\ 0 & 0 & 1 \end{pmatrix} U^{-1}
 \end{aligned}$$

Thus the probability that after at time n the spider and fly both at initial place is $M_{11}^n = 0.5(0.1^n + 0.46^n)$.

PROBLEM 4.(c):

Let $\tau = \min\{n; X_n = C\}$, $t_A = E[\tau|X_0 = A]$, $t_B = E[\tau|X_0 = B]$. t_A, t_B follows the linear equations:

$$\begin{aligned}
 t_A &= .28(1 + t_A) + .18(1 + t_B) + .54 \\
 t_B &= .18(1 + t_A) + .28(1 + t_B) + .54 \\
 \Rightarrow t_A &= t_B = 1.85
 \end{aligned}$$

The average duration is t_A .

PROBLEM 5.

Recall the formular $Var(X) = Var(E(X|Y)) + E(Var(X|Y))$. This formular holds because

$$\begin{aligned}
 &Var(E(X|Y)) + E(Var(X|Y)) \\
 &= E([E(X|Y)]^2) - [E(E(X|Y))]^2 + E(E(X^2|Y) - [E(X|Y)]^2) \\
 &= -[E(E(X|Y))]^2 + E(E(X^2|Y)) \\
 &= -[E(X)]^2 + E(X^2) \\
 &= Var(X)
 \end{aligned}$$

Given $X_0 = 1$, We have

$$\begin{aligned} Var(X_n) &= Var(E(X_n|X_{n-1})) + E(Var(X_n|X_{n-1})) \\ &= Var(X_{n-1} \cdot \mu) + E(X_{n-1} \cdot \sigma^2) \\ &\Rightarrow Var(X_n) = \mu^2 Var(X_{n-1}) + \sigma^2 \mu^{n-1} \end{aligned}$$

Thus we see the solution

$$Var(X_n) = \begin{cases} \sigma^2 \mu^{n-1} \frac{\mu^n - 1}{\mu - 1}, & \mu \neq 1 \\ n\sigma^2, & \mu = 1 \end{cases}$$

exactly satisfies the above recursive linear systems.