

SOLUTION FOR HOMEWORK 5, SEG5580

October 23, 2009

PROBLEM 1.

For state $(n_1, n_2, \dots, n_m)$ the limiting probability is $\frac{M!}{n_1!n_2!\dots n_m!} \frac{1}{m^M}$.
First,

$$\sum_{n_1+n_2+\dots+n_m=M} \frac{M!}{n_1!n_2!\dots n_m!} \frac{1}{m^M} = 1$$

Thus $\{\frac{M!}{n_1!n_2!\dots n_m!} \frac{1}{m^M}\}$ is probabilities.

Assume two states

$$(\dots n_k, \dots n_l, \dots) \triangleq \text{state i}$$

$$(\dots n_k - 1, \dots n_l + 1, \dots) \triangleq \text{state j}$$

The transition probability is $p_{ij} = \frac{n_k}{M} \frac{1}{m-1}$; $p_{ji} = \frac{n_l+1}{M} \frac{1}{m-1}$. If we define

$$\pi_i \triangleq \frac{M!}{n_1!\dots n_k!\dots n_l!\dots n_m!} \frac{1}{m^M}; \pi_j \triangleq \frac{M!}{n_1!\dots (n_k-1)!\dots (n_l+1)!\dots n_m!} \frac{1}{m^M}$$

Then it's obvious that

$$\pi_i p_{ij} = \pi_j p_{ji}$$

Then according to Theorem 1, the Markov chain is time reversible, with stationary probabilities $\{\frac{M!}{n_1!n_2!\dots n_m!} \frac{1}{m^M}\}$.

PROBLEM 2.

It's enough to show what happened when a new state "4" is added into the original states "1,2,3".

$$\begin{aligned}
 p_{12}p_{23}p_{34}p_{41} &\stackrel{?}{=} p_{14}p_{43}p_{32}p_{21} \\
 \Leftrightarrow p_{12}p_{23}p_{31}p_{13}p_{34}p_{41} &\stackrel{?}{=} p_{14}p_{43}p_{31}p_{13}p_{32}p_{21} \\
 \Leftrightarrow [p_{12}p_{23}p_{31}][p_{13}p_{34}p_{41}] &\stackrel{?}{=} [p_{14}p_{43}p_{31}][p_{13}p_{32}p_{21}]
 \end{aligned}$$

since the detailed balance equation holds for the case of three states, we see it also holds for the case of four states, and so on.

PROBLEM 3.

First, it's obvious that $\bar{\pi}_i$ satisfies $\bar{\pi}_i \geq 0$ and $\sum_i \bar{\pi}_i = 1$. Thus $\{\bar{\pi}_i\}$ is probabilities.

Then we define $\bar{p}_{ij}^* = \bar{p}_{ij}$. It's obvious that for $i \neq j$, we have

$$\bar{\pi}_i \bar{p}_{ij} = \bar{\pi}_j \bar{p}_{ji}^*$$

Thus according to Theorem 1, $\bar{\pi}_i$ is the stationary probabilities; and the truncated chain is time reversible.