

SOLUTION FOR HOMEWORK 7, SEG5580

November 4, 2009

PROBLEM 1.(a)

This is the birth-death model with parameter

$$\lambda_n = \begin{cases} n\lambda + \theta, & n \leq N-1 \\ n\lambda, & n \geq N \end{cases} \quad \mu_n = n\mu$$

PROBLEM 1.(b)

$$\begin{aligned} p_1 &= \frac{\lambda_0}{\mu_1} p_0 \\ p_2 &= \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} p_0 \\ p_3 &= \frac{\lambda_0 \lambda_1 \lambda_2}{\mu_1 \mu_2 \mu_3} p_0 \\ &\dots \end{aligned} \tag{1}$$

The proportion of time that immigration is unrestricted is

$$\begin{aligned} &\frac{p_0 + p_1 + p_2}{p_0 + p_1 + p_2 + \dots} \\ &= \frac{1 + \frac{1}{2} + \frac{1 \cdot 2}{2 \cdot 4}}{1 + \frac{1}{2} + \frac{1 \cdot 2}{2 \cdot 4} + \frac{1 \cdot 2 \cdot 3}{2 \cdot 4 \cdot 6} + \frac{1 \cdot 2 \cdot 3 \cdot 4}{2 \cdot 4 \cdot 6 \cdot 8} + \dots} \\ &= \frac{1 + \frac{1}{2} + \frac{1 \cdot 2}{2 \cdot 4}}{1 + \frac{1}{2} + \frac{1 \cdot 2}{2 \cdot 4} + \frac{1 \cdot 2 \cdot 3}{2 \cdot 4 \cdot 6} + 3[(\frac{1}{2})^4 \frac{1}{4} + (\frac{1}{2})^5 \frac{1}{5} + (\frac{1}{2})^6 \frac{1}{6} + \dots]} \end{aligned}$$

Simple calculus shows

$$\left(\frac{1}{2}\right)^4 \frac{1}{4} + \left(\frac{1}{2}\right)^5 \frac{1}{5} + \left(\frac{1}{2}\right)^6 \frac{1}{6} + \dots = -(0.5^3/3 + 0.5^2/2 + 0.5) - \ln(1 - 0.5) = 0.02648$$

Thus $\frac{p_0 + p_1 + p_2}{p_0 + p_1 + p_2 + \dots} = 0.8954$; And The proportion of time that immigration is restricted is $1 - 0.8954 = 0.1046$.

PROBLEM 2.(a)

This is the M/M/1 queue model truncated at 2. $\lambda = 3, \mu = 4$. The stationary distribution for original M/M/1 queue is $p_n \propto (\lambda/\mu)^n = (3/4)^n$. And the stationary distribution for M/M/1 queue truncated at 2 is

$$p_1^A = \frac{(3/4)}{(3/4) + (3/4)^2} = 0.57, p_2^A = \frac{(3/4)^2}{(3/4) + (3/4)^2} = 0.43$$

Thus the average number of customers is $p_1^A + 2p_2^A = 1.43$

PROBLEM 2.(b)

In the original untruncated queue, a typical customer come, and he see i customers already in the room with probability $p_i; i = 0, 1, 2, \dots$. Only when the customer see 0 or 1 customer in the room, he will enter the room in the truncated queue. Thus the proportion of potential customers get served is

$$\begin{aligned} & \frac{p_0 + p_1}{\sum_{i \geq 0} p_i} \\ &= \frac{(3/4)^0 + (3/4)^1}{(3/4)^0 + (3/4)^1 + (3/4)^2 + \dots} \\ &= 0.4375 \end{aligned}$$

PROBLEM 2.(c)

If $\mu = 8$

$$p_1^A = \frac{(3/8)}{(3/8) + (3/8)^2} = 0.73, p_2^A = \frac{(3/8)^2}{(3/8) + (3/8)^2} = 0.27$$

And the proportion of potential customers get served is

$$\frac{(3/8)^0 + (3/8)^1}{(3/8)^0 + (3/8)^1 + (3/8)^2 + \dots} = 0.8594$$

So the barber does $0.8594/0.4375 - 1 = 96/100$ more business.

PROBLEM 3.

Assume the infinitesimal rates are $q_{i_1 j_1}^X$ for $\{X_t, t \geq 0\}$, $q_{i_2 j_2}^Y$ for $\{Y_t, t \geq 0\}$. Since $\{X_t, t \geq 0\}$ and $\{Y_t, t \geq 0\}$ are independent, the infinitesimal rates for $\{(X_t, Y_t), t \geq 0\}$ is $q_{i_1 j_1}^X q_{i_2 j_2}^Y$. Since $\{X_t, t \geq 0\}$ and $\{Y_t, t \geq 0\}$ both are time-reversible, the two detailed balance equation holds

$$p_{i_1}^X q_{i_1 j_1}^X = p_{j_1}^X q_{j_1 i_1}^X$$

$$p_{i_2}^Y q_{i_2 j_2}^Y = p_{j_2}^Y q_{j_2 i_2}^Y$$

Then we get the third detailed balance equation for $\{(X_t, Y_t), t \geq 0\}$

$$[p_{i_1}^X p_{i_2}^Y] q_{i_1 j_1}^X q_{i_2 j_2}^Y = [p_{j_1}^X p_{j_2}^Y] q_{j_1 i_1}^X q_{j_2 i_2}^Y$$

And the probabilities $p_{i_1}^X p_{i_2}^Y$ satisfies

$$\sum_{i_1 i_2} p_{i_1}^X p_{i_2}^Y = 1$$

Then we see $\{(X_t, Y_t), t \geq 0\}$ is time-reversible.

PROBLEM 4.

This is the truncated two-dimensional M/M/1 queues. For the original un-truncated two-dimensional M/M/1 queues, since the two queues are independent, the stationary distributions

$$p(n, m) \propto (\lambda_1/\mu_1)^n (\lambda_2/\mu_2)^m$$

Thus for the truncated two-dimensional M/M/1 queues, the stationary distributions

$$p(n, m) = \frac{(\lambda_1/\mu_1)^n (\lambda_2/\mu_2)^m}{\sum_{i+j \leq N} (\lambda_1/\mu_1)^i (\lambda_2/\mu_2)^j}$$