

SOLUTION FOR HOMEWORK 9, SEG5580

December 7, 2009

PROBLEM 1.

$P(X_{N_t+1} \geq x) = P(\beta_t \geq x) = K_x(t).$
 $K_x(t) = 1 - F(x \vee t) + \int_0^t k_x(t-y) dF(y).$
 $K_x(t) = 1 - F(x \vee t) + \int_0^t 1 - F((x \vee t) - y) dm(y).$
 For $t \leq x$, $K_x(t) = 1 - F(x) + \int_0^t 1 - F(x \vee (t-y)) dm(y) \geq 1 - F(x).$
 For $t > x$, $K_x(t) = 1 - F(t) + \int_0^t 1 - F(x \vee (t-y)) dm(y)$
 $= 1 - F(t) + \int_0^{t-x} 1 - F(t-y) dm(y) + \int_{t-x}^t 1 - F(x) dm(y)$
 $\triangleq LHS$, and we define $RHS \triangleq 1 - F(x).$
 When $x = 0$, we have $LHS = 1 = RHS$, and for $x > 0$,
 $\frac{d}{dx} LHS = -(1 - F(x))m'(t-x) + (1 - F(x))m'(t-x) = 0$ and
 $\frac{d}{dx} RHS = -F'(x) \leq 0.$
 In addition, when $x \rightarrow t$, $LHS \rightarrow 1 - F(t) + \int_0^t 1 - F(t) dm(y)$ and $RHS \rightarrow 1 - F(t).$ Based on these results, we know that $LHS \geq RHS$, which is the conclusion to be proved.
 When $F(x) = 1 - e^{-\lambda x}$, by the memoryless property of exponential distribution, we know that $P[X_{N(t)+1} \geq x] = 1 - F(x) = e^{-\lambda x}.$

PROBLEM 2.

By the Central Limit Theorem (Theorem 5), we have $Var[\frac{N(t)-t/\mu}{\sigma\sqrt{t/\mu^3}}] \rightarrow 1.$
 On the other hand, $Var[\frac{N(t)-t/\mu}{\sigma\sqrt{t/\mu^3}}] = \frac{Var[N(t)]}{\sigma^2 t/\mu^3}$, then we have the conclusion
 $\frac{Var[N(t)]}{t} \rightarrow \frac{\sigma^2}{\mu^3}.$

PROBLEM 3.

Denote X_n as the number of hours spending in the n th choice of rooms. Then, X_n 's are i.i.d. with the distribution: $P(X_n = 2) = P(X_n = 4) = P(X_n = 1) = 1/3$. Denote N as $N \triangleq \inf\{n \geq 1 : X_n = 2\}$, then N is a stopping time and is geometrically distributed with $E[N] = 3$. So, by the Wald Equation, we have $E[T] = E[X]E[N] = 7$.

PROBLEM 4.

$$\lim_{t \rightarrow \infty} P(\text{process is in state } i \text{ at time } t) = \frac{E[X_i]}{\sum_{j=1}^N E[X_j]}.$$

PROBLEM 5.

$$(a). m_D(t) = \int_0^t 1 + m(t-y)dG(y) = P(X_1 \leq t) + \int_0^t m(t-y)dG(y),$$

$$\frac{m_D(t)}{t} = \frac{P(X_1 \leq t)}{t} + \frac{1}{t} \int_0^t m(t-y)dG(y).$$

Since $\frac{m(t-y)}{t} \rightarrow \frac{1}{\mu}$ and $\int_0^t dG(y) \rightarrow 1$, simple analysis will show that $\frac{1}{t} \int_0^t m(t-y)dG(y) \rightarrow \frac{1}{\mu}$.

$$(b). P(Y_D(t) \geq x) = \int_0^t P(\gamma(t-y) \geq x)dG(y) + \int_{t+x}^{+\infty} dG(y).$$

$$\lim_{t \rightarrow \infty} P(Y_D(t) \geq x) = \lim_{t \rightarrow \infty} P(\gamma(t) \geq x) = \frac{1}{\mu} \int_x^{+\infty} (1 - F(y))dy.$$

PROBLEM 6.

$$E[R_{N(t)+1}] = \begin{cases} E[R_1|X_1 = x], & x > t \\ E[R_{N(t-x)+1}], & x \leq t \end{cases}$$

$$\begin{aligned} A(t) &\triangleq E[R_{N(t)+1}] = \int_t^\infty E[R_1|X_1 = x]dF(x) + \int_0^t E[R_{N(t-x)+1}]dF(x) \\ &= \int_t^\infty E[R_1|X_1 = x]dF(x) + \int_0^t A(t-x)dF(x) \\ &= a(t) + \int_0^t A(t-x)dF(x), \text{ where } a(t) \triangleq \int_t^\infty E[R_1|X_1 = x]dF(x). \\ \lim_{t \rightarrow \infty} A(t) &= \frac{1}{E[X_1]} \int_0^\infty a(t)dt, \text{ and} \\ \int_0^\infty a(t)dt &= \int_0^\infty dt \int_t^\infty E[R_1|X_1 = x]dF(x) \\ &= \int_t^\infty dF(x) \int_0^x E[R_1|X_1 = x]dt = \int_0^\infty x E[R_1|X_1 = x]dF(x) = E[R_1 X_1]. \end{aligned}$$

So, we have $\lim_{t \rightarrow \infty} A(t) = \frac{E[R_1 X_1]}{E[X_1]}$.