

Lecture 6: Martingales II

0.1 Martingales

- A stochastic process $\{Z_n, n \geq 1\}$ is said to be a *martingale* process if $E[|Z_n|] < +\infty$ and

$$E[Z_n | Z_1, \dots, Z_{n-1}] = Z_{n-1}.$$

for all n .

Example 1 Let $X_1, X_2, \dots$ be independent random variables with mean 0; and let $Z_n = \sum_{i=1}^n X_i$. Then, $\{Z_n, n \geq 1\}$ is a martingale. Let $X_1, X_2, \dots$ be independent random variables with mean 1; and let $Z_n = \prod_{i=1}^n X_i$. Then, $\{Z_n, n \geq 1\}$ is a martingale.

Example 2 Let $Y, X_1, X_2, \dots$ be arbitrary random variables such that $E[|Y|] < +\infty$; and let $Z_n = E[Y | X_1, \dots, X_n]$. $\{Z_n, n \geq 1\}$ is a martingale. This martingale is often called a Doob type martingale.

Example 3 For any random variables $X_1, X_2, \dots$, the random variable $X_i - E[X_i | X_1, \dots, X_{i-1}]$ have mean 0. Even though they need not be independent, their partial sums constitutes a martingale. That is,

$$Z_n = \sum_{i=1}^n \{X_i - E[X_i | X_1, \dots, X_{i-1}]\}$$

is a martingale.

0.2 Stopping Times

- A positive integer-valued, possibly infinite, random variable N is said to be a *stopping time* for the process $\{Z_n, n \geq 1\}$ if $P[N < +\infty] = 1$ and the event $\{N = n\}$ is determined by the random variables $Z_1, \dots, Z_n$ for all n . Let

$$\bar{Z}_n = \begin{cases} Z_n, & \text{if } n \leq N; \\ Z_N, & \text{if } n > N. \end{cases}$$

$\{\bar{Z}_n\}$ is called the *stopped* process.

Theorem 1 If N is a stopping time for the martingale $\{Z_n\}$, then the stopped process $\{\bar{Z}_n\}$ is also a martingale.

Theorem 2 (Optional Sampling Theorem) If either:

- (i). $\{\bar{Z}_n\}$ are uniformly bounded, or;
- (ii). N is bounded, or;
- (iii). $E[N] < +\infty$, and there is an $M < +\infty$ such that

$$E[|Z_{n+1} - Z_n| | Z_1, \dots, Z_n] < M,$$

then, $E[Z_N] = E[Z_1]$.

Theorem 3 (Wald Equation) If $X_i, i \geq 1$, are independent and identically distributed with $E[|X|] < +\infty$ and if N is a stopping time for $X_1, X_2, \dots$ with $E[N] < +\infty$, then

$$E\left[\sum_{i=1}^N X_i\right] = E[N]E[X].$$

Example 4 Consider an individual who starts at position 0 and at each step either moves 1 position to the right with prob. p or 1 to the left with prob. $1 - p$. Assume that the successive movements are independent. If $p > 1/2$ find the expected number of step it takes until the individual reaches position $i, i > 0$.

Example 5 (Gambler ruin problem revisited) A gambler plays a fair game with equal chance of winning \$1 or losing \$1 each round. His initial wealth is \$ a and decides to leave the game if he wins up to \$ $(b + a)$ or loses all money he has initially. What is the chance he loses all the money when stopping? What is the expected number of rounds until he stops?

0.3 Submartingales and Supermartingales

- A stochastic process $\{Z_n, n \geq 1\}$ having $E[|Z_n|] < +\infty$ is said to be a *submartingale* process if

$$E[Z_n | Z_1, \dots, Z_{n-1}] \geq Z_{n-1}.$$

for all n and is said to be a *submartingale* process if

$$E[Z_n | Z_1, \dots, Z_{n-1}] \leq Z_{n-1}.$$

for all n .

Theorem 4 If N is a stopping time for $\{Z_n, n \geq 1\}$ such that any one of the sufficient conditions of the optional sampling theorem is satisfied, then

$$\begin{aligned} E[Z_N] &\geq E[Z_1] \quad \text{for a submartingale;} \\ E[Z_N] &\leq E[Z_1] \quad \text{for a supermartingale.} \end{aligned}$$

Theorem 5 If $\{Z_n, n \geq 1\}$ is a martingale and f a convex function, then $\{f(Z_n), n \geq 1\}$ is a submartingale.

Theorem 6 (Kolmogorov's maximal inequality for submartingales) If $\{Z_n, n \geq 1\}$ is a nonnegative submartingale, then, for $a > 0$,

$$P[\max\{Z_1, \dots, Z_n\} > a] \leq \frac{E[Z_n]}{a};$$

Corollary 1 If $\{Z_n, n \geq 1\}$ is a martingale, then, for $a > 0$, both inequalities hold:

$$\begin{aligned} P[\max\{|Z_1|, \dots, |Z_n|\} > a] &\leq \frac{E[|Z_n|]}{a}; \\ P[\max\{|Z_1|, \dots, |Z_n|\} > a] &\leq \frac{E[Z_n^2]}{a^2}. \end{aligned}$$

0.4 The Martingale Convergence Theorem

- Given a submartingale $\{X_n\}$, real numbers $a < b$, and a positive integer N , define $V_{a,b}$ to be the number of pairs (i, j) with $0 \leq i < j \leq N$, for which $X_i \leq a, a < X_k < b$ for $i < k < j$ and $X_j \geq b$. Then $V_{a,b}$ counts the number of crosses from a level below a to a level above b . Since submartingales have an up-trend, we expect an up bound on the number of $V_{a,b}$. That is the upcrossing inequality we are going to discuss as follows.

Define

$$\hat{X}_n = (X_n - a)^+.$$

$\hat{X}_n$ is also a submartingale because the function $(x - a)^+$ is convex. Define $T_1 = 0$ and for $k = 1, \dots, N$, set

$$T_k = \begin{cases} N, & \text{if } \hat{X}_j \neq 0, \text{ for } j > T_{k-1}; \\ \min\{j : j > T_{k-1}, \hat{X}_j = 0\}, & \text{otherwise.} \end{cases}$$

if k is even, and

$$T_k = \begin{cases} N, & \text{if } \hat{X}_j < b - a, \text{ for all } j > T_{k-1}; \\ \min\{j : j > T_{k-1}, \hat{X}_j \geq b - a\}, & \text{otherwise.} \end{cases}$$

for k is odd. Then each T_k is a stopping time. Using Theorem 4, we have

$$E[\hat{X}_{T_k}] \leq E[\hat{X}_{T_{k+1}}]. \quad (1)$$

Furthermore,

$$\begin{aligned} \hat{X}_N - \hat{X}_0 &= \sum_{k=1}^N (\hat{X}_{T_{k+1}} - \hat{X}_{T_k}) \\ &= \sum_{k=2,4,\dots} (\hat{X}_{T_{k+1}} - \hat{X}_{T_k}) + \sum_{k=1,3,\dots} (\hat{X}_{T_{k+1}} - \hat{X}_{T_k}). \end{aligned}$$

Now, if k is even, $\hat{X}_{T_{k+1}} - \hat{X}_{T_k}$ is nonzero only if an upcrossing occurs and then is at least $(b - a)$, while the expected value of the second sum is nonnegative by (1). Thus we have

Theorem 7 (Upcrossing Inequality for submartingales)

$$E[V_{a,b}] \leq \frac{E[(X_N - a)^+] - E[(X_0 - a)^+]}{b - a}.$$

- Theorem 8 (The Martingale Convergence Theorem)** If X_n is a submartingale with $\sup_n E[X_n^+] < +\infty$, then, as $n \rightarrow +\infty$, X_n converges to a limit X almost surely with $E[|X|] < +\infty$.

Example 6 Let $Y_1, Y_2, \dots$ be nonnegative i.i.d. random variables with $E[Y_m] = 1$ and $P[Y_m = 1] < 1$. $X_n = \prod_{m=1}^n Y_m$ defines a martingale. According to the convergence theorem, X_n has a limit. We can further show that the limit should be $X \equiv 0$.

Example 7 Let $\{X_n, n \geq 0\}$ be an irreducible recurrent Markov process with finite state space and a transition probability matrix $\mathbf{P} = [p_{ij}]$. Suppose that a bounded vector $\mathbf{y}$ satisfies

$$\mathbf{y} = \mathbf{P}\mathbf{y}.$$

Then, $\mathbf{y}$ is a constant vector.

Homework (Due on Dec. 9):

1. For a martingale $\{Z_n, n \geq 1\}$, let $X_i = Z_i - Z_{i-1}, i \geq 1$, where $Z_0 = 0$. Show that

$$\text{Var}(Z_n) = \sum_{i=1}^n \text{Var}(X_i).$$

2. Consider a Markov chain S_n which at each transition either goes up 1 with probability p or down 1 with probability $q = 1 - p$. Argue that $(q/p)^{S_n}$ is a martingale.
3. Consider a Markov chain X_n with a finite state space $\{1, 2, \dots, N\}$ and $p_{N,N} = 1$. Let $f(i)$ be the probability that this chain eventually enters state N given that it starts in state i . Show that $\{f(X_n), n \geq 0\}$ is a martingale. Does the limit of $f(X_n)$ exist? If so, what is it? If not, why not?
4. Let $Z_n = \prod_{i=1}^n X_i$ where $X_i, i \geq 1$ are independent random variables with

$$P[X_i = 2] = P[X_i = 0] = 1/2.$$

Let $N = \min\{n : Z_n = 0\}$. Is the optional sampling theorem applicable? If so, what would you conclude? If not, why not?