

Lecture 7: Martingales II

0.1 The Martingale Mean-Square Convergence Theorem

- **Theorem 1** Let $\{X_n\}$ be a martingale satisfying $\sup_n E[X_n^2] \leq K$ for some constant K . Then $\{X_n\}$ converges as $n \rightarrow \infty$ to a limit random variable X both with probability 1 and in mean square sense. That is,

$$P\left[\lim_{n \rightarrow +\infty} X_n = X\right] = 1$$

and

$$\lim_{n \rightarrow +\infty} E[|X_n - X|^2] = 0.$$

Example 1 (An Urn Model) Consider an urn containing n red and m green balls. A ball is drawn at random from the urn and it and one more of the same color are then returned. The experiment is repeated indefinitely. Let X_k be the fraction of red balls at stage k . Then X_k is a martingale. It is bounded and therefore the limit X exists in both senses of probability 1 and mean square.

We will derive the distribution of X as follows. Let $Y_k = (n + m + k)X_k$ be the total number of red balls at stage k . Then

$$P[Y_k = i] = \frac{\binom{i-1}{n-1} \binom{N-i-1}{m-1}}{\binom{N-1}{n+m-1}}$$

where $n \leq i \leq n + k$, $N = n + m + k$. Using Stirling approximation $M! \sim e^{-M} M^M \sqrt{2\pi M}$, we can show that

$$\binom{M}{j} \sim (M-j)^j / j!.$$

Then

$$\begin{aligned} P[X_k \leq x] &= P[Y_k \leq Nx] \\ &= \sum_{i=0}^{Nx} \binom{i-1}{n-1} \binom{N-i-1}{m-1} / \binom{N-1}{n+m-1} \\ &\approx \frac{(n+m-1)!}{(n-1)!(m-1)!} \sum_{i=0}^{Nx} \left(\frac{i-n}{k}\right)^{n-1} \left(\frac{k-i+n}{k}\right)^{m-1} \frac{1}{k} \\ &= \frac{\Gamma(n+m)}{\Gamma(n)\Gamma(m)} \sum_{i=0}^{Nx} \left(\frac{i}{k} - \frac{n}{k}\right)^{n-1} \left(1 - \frac{i}{k} + \frac{n}{k}\right)^{m-1} \frac{1}{k} \\ &\rightarrow \frac{\Gamma(n+m)}{\Gamma(n)\Gamma(m)} \int_0^x z^{n-1} (1-z)^{m-1} dz. \end{aligned}$$

This is the Beta distribution.

Example 2 (Branching Processes) Let Y_n be a branching process and suppose that the mean of the progeny distribution is $m < +\infty$. Then $X_n = Y_n/m^n$ is a martingale. We know that

$$\text{Var}[Y_n^2] = \frac{\sigma^2}{m} \left(\frac{m^{2n} - m^n}{m - 1} \right)$$

and use this to get

$$\begin{aligned} E[X_n^2] &= \{\text{Var}[Y_n^2] + E[Y_n^2]\}/m^{2n} \\ &= \frac{\sigma^2}{m} \left(\frac{m^{2n} - m^n}{m - 1} \right) + 1 \\ &\leq \frac{\sigma^2}{m(m - 1)} + 1. \end{aligned}$$

Thus, X_n converges in both senses of probability 1 and mean square. We have $E[X] = X_0 = 1$. Y_n asymptotically grows exponentially fast at rate m .

0.2 Azuma's Inequality for Martingales

- **Theorem 2** Let $Z_n, n \geq 1$ be a martingale with mean $\mu = E[Z_n]$. Let $Z_0 = \mu$ and suppose that for nonnegative constants $\alpha_i, \beta_i, i \geq 1$, $-\alpha_i \leq Z_i - Z_{i-1} \leq \beta_i$. Then for any $n \geq 0, a > 0$:

$$P[Z_n - \mu \geq a] \leq \exp(-2a^2 / \sum_{i=1}^n (\alpha_i + \beta_i)^2);$$

(ii).

$$P[Z_n - \mu \leq -a] \leq \exp(-2a^2 / \sum_{i=1}^n (\alpha_i + \beta_i)^2);$$

Example 3 Let $X_1, \dots, X_n$ be random variables such that $E[X_1] = 0$ and $E[X_i | X_1, \dots, X_{i-1}] = 0$. Then $\sum_{i=1}^j X_i$ is a zero-mean martingale. If $-\alpha_i \leq X_i \leq \beta_i$ for all i , then

$$P[\sum_{i=1}^j X_i \geq a] \leq \exp(-2a^2 / \sum_{i=1}^n (\alpha_i + \beta_i)^2);$$

and

$$P[\sum_{i=1}^j X_i \leq -a] \leq \exp(-2a^2 / \sum_{i=1}^n (\alpha_i + \beta_i)^2).$$

Example 4 Suppose that n balls are put in m urns in such a manner that each ball is equally likely to go into any of the urns. We can use the Azuma's inequality to study the tail probability of X , the number of empty urns.

$$X = \sum_{i=1}^m \mathbf{1}_{\{\text{urn } i \text{ is empty}\}}$$

and thus,

$$E[X] = m(1 - 1/m)^n = \mu.$$

Now denote X_j to be the urn in which the j th ball is placed, $j = 1, \dots, n$. Let $Z_0 = E[X]$ and $Z_i = E[X|X_1, \dots, X_i]$.

$$E[X|X_1, \dots, X_{i-1}] = (m - D)(1 - 1/m)^{n-i+1},$$

where D is the number of urns having at least one ball after the first $i - 1$ balls are distributed. On the other hand,

$$E[X|X_1, \dots, X_i] = \begin{cases} (m - D)(1 - 1/m)^{n-i} & \text{if } X_i \notin (X_1, \dots, X_{i-1}); \\ (m - D - 1)(1 - 1/m)^{n-i} & \text{if } X_i \in (X_1, \dots, X_{i-1}). \end{cases}$$

Hence, the two possible values of $Z_i - Z_{i-1}$ are

$$\frac{m - D}{m}(1 - 1/m)^{n-i} \quad \text{and} \quad \frac{-D}{m}(1 - 1/m)^{n-i}.$$

Since $1 \leq D \leq \min(i - 1, m)$, we thus obtain that $-\alpha_i \leq Z_i - Z_{i-1} \leq \beta_i$ with

$$\alpha_i = \min\left(\frac{i - 1}{m}, 1\right)(1 - 1/m)^{n-i} \quad \text{and} \quad \beta_i = (1 - 1/m)^{n-i+1}.$$

From Azuma's inequality, we have

$$P[X - \mu \geq a] \leq \exp(-2a^2 / \sum_{i=2}^n (\alpha_i + \beta_i)^2)$$

and

$$P[X - \mu \leq -a] \leq \exp(-2a^2 / \sum_{i=1}^n (\alpha_i + \beta_i)^2).$$

Corollary 1 Let h be a function such that if the vectors $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$ differ in at most one coordinate then $|h(\mathbf{x}) - h(\mathbf{y})| \leq 1$. Let $(X_1, \dots, X_n)$ be independent random variables. Then, with $\mathbf{X} = (X_1, \dots, X_n)$, we have for $a > 0$ that

(i).

$$P[h(\mathbf{X}) - E[h(\mathbf{X})] \geq a] \leq \exp(-a^2/2n);$$

(ii).

$$P[h(\mathbf{X}) - E[h(\mathbf{X})] \leq -a] \leq \exp(-a^2/2n);$$

Homework (No Submission Required):

1. Let X denote the number of heads in n independent flips of a fair coin. Show that

$$P[X - n/2 \geq a] \leq \exp(-2a^2/n);$$

and

$$P[X - n/2 \leq -a] \leq \exp(-2a^2/n).$$

2. Let X_n be a Markov process for which X_0 is uniform on $(0, 1)$, and conditional on X_n ,

$$X_{n+1} = \begin{cases} \alpha X_n + 1 - \alpha & \text{with probability } X_n; \\ \alpha X_n & \text{with probability } 1 - X_n. \end{cases}$$

where $0 < \alpha < 1$. Discuss the limiting properties of the sequence of X_n .

3. An urn initially contains one white and one black ball. At each stage a ball is drawn and is then replaced in the urn along with another ball of the same color. Let Z_n denote the fraction of balls in the urn that are white after the n th replication.
- (a). Show that $\{Z_n, n \geq 1\}$ is a martingale;
 - (b). Show that the probability that the fraction of white balls in the urn is ever as large as $3/4$ is at most $2/3$.
4. Consider a sequence of independent tosses of a coin and let $P[\text{head}]$ be the probability of a head on any toss. Let A be the hypothesis that $P[\text{head}] = a$ and B be the hypothesis that $P[\text{head}] = b, a \neq b$. Let X_i denote the outcome of the i th toss and let

$$Z_n = \frac{P[X_1, \dots, X_n | \text{Hypothesis A is true}]}{P[X_1, \dots, X_n | \text{Hypothesis B is true}]}.$$

Show that if B is true, then

- (a). Z_n is a martingale, and
- (b). $\lim_{n \rightarrow +\infty} Z_n$ exists with probability 1.
- (c). If $b \neq a$, what is $\lim_{n \rightarrow +\infty} Z_n$?