

Lecture 2: Poisson processes

0.1 Exponential Distribution and Memoryless

- A continuous random variable X is said to have an *exponential distribution* with parameter $\lambda > 0$ if its probability density function is given by

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0.$$

- Moment generating functions: $E[e^{tX}] = \lambda/(\lambda - t)$. $E[X] = 1/\lambda$ and $V(X) = 1/\lambda^2$.
- Memoryless property:

$$P[X > s + t | X > t] = P[X > s].$$

0.2 Definition of Poisson Processes

- A *Poisson process* is named after the French mathematician Simeon-Denis Poisson (1781—1840). It has wide applications in stochastic modeling. Examples that are well-modeled as Poisson processes include radioactive decay of atoms, telephone calls arriving at a switchboard, page view requests to a website, and rainfall.
- Assume that $N(t)$ represents the total number of ‘telephone calls’ that have occurred up to time t . It must satisfy
 - $N(t) \geq 0$ and integer-valued;
 - $N(s) \leq N(t)$ for all $s < t$;
 - $N(t) - N(s)$ equals the number of ‘telephone calls’ that have occurred in the interval $(s, t]$.

In the probability theory, such $\{N(t), t \geq 0\}$ is said to be a *counting process*. A counting process is said to possess the *independent increment* property if the numbers of ‘telephone calls’ that occur in disjoint time intervals are independent. A counting process is said to possess the *stationary increment* property if the numbers of ‘telephone calls’ that occur in any time interval depends only on the length of the time interval.

- A counting process $\{N(t), t \geq 0\}$ is said to be a *Poisson process having rate λ* if
 - $N(0) = 0$;
 - The process has independent increments;
 - The number of events in any interval of length t is Poisson distributed with mean λt . That is, for all $s, t \geq 0$,

$$P[N(t+s) - N(s) = n] = e^{-\lambda t} \frac{(\lambda t)^n}{n!}.$$

0.3 Interarrival and Waiting Time Distributions

- Let X_1 be the arrival times of the first event and X_n be the time between the $(n-1)$ st and the n th event. The sequence $\{X_n, n \geq 1\}$ is called the sequence of *interarrival times*.

Theorem 1 For any $t > 0$,

$$P[X_1 > t] = e^{-\lambda t}$$

and

$$P[X_{i+1} > t | X_1 + X_2 + \cdots + X_i = s] = e^{-\lambda t}.$$

In words, $\{X_n, n \geq 1\}$ is a sequence of independent identically distributed exponential random variables having mean $1/\lambda$.

- Another quantity of interest is S_n , the arrival time of the n th event, also called the *waiting time* until the n th event.

Theorem 2 The probability distribution function of S_n

$$F(t) = P[S_n \leq t] = \sum_{j=n}^{+\infty} e^{-\lambda t} \frac{(\lambda t)^j}{j!}$$

and the probability density function of S_n is

$$f(t) = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}.$$

In distribution, $S_n = X_1 + \cdots + X_n$, a sum of interarrival times up to n . It is known as the *Erlang distribution*, which was developed by Agner Krarup Erlang (1878 — 1929), a Danish mathematician and engineer, to examine the number of telephone calls which might be made in a period of time to the operators of the switching stations.

0.4 Conditional Distribution of the Arrival Times

- Let $Y_1, \dots, Y_n$ be n random numbers. We say that $Y_{(1)}, \dots, Y_{(n)}$ are the *order statistics* corresponding to $Y_1, \dots, Y_n$ if the $Y_{(k)}$ is the k th smallest value among $Y_1, \dots, Y_n$.

Theorem 3 When $Y_1, \dots, Y_n$ are independent and identically distributed continuous random variables with probability density f . Then the joint density of the order statistics $Y_{(1)}, \dots, Y_{(n)}$ is given by

$$f(y_1, \dots, y_n) = n! \prod_{i=1}^n f(y_i), \quad y_1 < \cdots < y_n.$$

In particular, if Y_i 's are uniformly distributed over $(0, t)$, then it follows that

$$f(y_1, \dots, y_n) = \frac{n!}{t^n}, \quad y_1 < \cdots < y_n.$$

- Consider the conditional distribution of the arrival times of a Poisson process.

Theorem 4 Given $N(t) = n$, the n arrival times $S_1, \dots, S_n$ have the same distribution as the order statistics corresponding to n independent random variables uniformly distributed on the interval $(0, t)$.

Example 1 Suppose that travelers arrive at the University Railway Station in accordance with a Poisson process with rate λ . If the train comes at time t , what is the expected sum of the waiting times of travelers arriving in $(0, t)$?

Example 2 Suppose that a device is subject to shocks that occur in accordance with a Poisson process having a rate λ . The i th shock gives rise to a damage D_i . The D_i are assumed to be independent and identically distributed and also independent of the Poisson process. The damage due to a shock is assumed to decrease exponentially in time. That is, if a shock has an initial damage D , then a time t later its damage will be De^{-at} . What is $E[D(t)]$, the expected total damage at time t ?

0.5 Stochastic Thinning and Superposition

- Suppose that each event of a Poisson process is classified as being either type-I or type-II events. Furthermore, assume that if an event is classified as being a type-I with probability p , independent of all else. Let $N_i(t)$, $i = 1, 2$ represent the numbers of type- i events occurring by t . Then, $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ are independent Poisson processes with rate $p\lambda$ and $(1 - p)\lambda$, respectively.

We may extend the preceding statement into a time-inhomogeneous setting. Suppose that if an event occurs at time s , it is classified as being a type-I event with probability $P(s)$. Then, for any given t , $N_1(t)$ and $N_2(t)$ are independent Poisson random numbers having respective means $p\lambda t$ and $(1 - p)\lambda t$, where

$$p = \frac{1}{t} \int_0^t P(s) ds.$$

Example 3 Suppose that customers arrive at a service station in accordance with a Poisson process at rate λ . Upon arrival the customer is immediately served by one of an infinite number of possible servers, and the service times are assumed to be independent with a common distribution G . What is the joint distribution of the number of customers that have completed their service and the number that are in service at time t ?

- The superposition of two Poisson processes with intensities λ_1 and λ_2 is a Poisson process with intensity $\lambda = \lambda_1 + \lambda_2$ (Homework question).

0.6 Alternative Definition of Poisson Processes

- The function f is said to be $o(h)$ if

$$\lim_{h \rightarrow 0} \frac{f(h)}{h} = 0.$$

- An alternative definition of Poisson processes:

Theorem 5 A counting process $\{N(t), t \geq 0\}$ is said to be a Poisson process having rate λ if

- $N(0) = 0$;
- The process has independent and stationary increments;
-

$$P[N(t+h) - N(t) \geq 2] = o(h)$$

and

$$P[N(t+h) - N(t) = 1] = \lambda h + o(h).$$

Homework (Due on Sept. 21):

1. Let $\{N(t), t \geq 0\}$ be a Poisson process with rate λ . Calculate $E[N(t) \cdot N(t+s)]$.
2. Suppose that $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ are independent Poisson processes with rates λ_1 and λ_2 . Show that $\{N_1(t) + N_2(t), t \geq 0\}$ is a Poisson process with rate $\lambda_1 + \lambda_2$.
3. Cars pass a certain street corner according to a Poisson process with rate λ . A person wanting to cross the street waits until she can see that no cars will come by in the next T time units. Find the expected time that the person waits before starting to cross.
4. Buses arrive at a certain stop according to a Poisson process with rate λ . If you take the bus from that stop to home then it takes a time R . If you walk, it takes a time W to arrive home. Suppose that your policy when arriving at the bus stop is to wait up to a time s , and if a bus has not yet arrived by that time then you walk home.
 - (a). Compute the expected time from when you arrive at the bus stop until you reach home.
 - (b). Show that if $W < 1/\lambda + R$ then the expected time of (a) is minimized by letting $s = 0$ (that is, you will never wait for a bus); if $W > 1/\lambda + R$ then it is minimized by letting $s = +\infty$ (that is, you continue to wait for the bus); and when $W = 1/\lambda + R$ then all values of s give the same expected time.
5. Let $X_1, \dots, X_n$ be independent continuous random variables with common density function f . Let $X_{(i)}$ denote the i th smallest of $X_1, \dots, X_n$.
 - (a). Note that in order for $X_{(i)}$ to equal x , exactly $i - 1$ of the X 's must be less than x , one must equal x , and the other $n - i$ must all be greater than x . Using this fact argue that the density function of $X_{(i)}$ is given by

$$f_{X_{(i)}}(x) = \frac{n!}{(i-1)!(n-i)!} (F(x))^{i-1} (1 - F(x))^{n-i} f(x).$$

- (b). $X_{(i)}$ will be less than x if and only if how many of the the X 's are less than x ?
- (c). Use (b) to obtain an expression for $P[X_{(i)} \leq x]$.
- (d). Let S_i denote the time of the i th event of Poisson process $\{N(t), t \geq 0\}$. Find $E[S_i | N(t) = n]$ for $i < n$.

6. Suppose that cars enter a one-way infinite highway at a Poisson rate λ . The i th car to enter chooses a velocity V_i and travels at this velocity constantly afterwards. Assume that the V_i 's are independent positive random variables having a common distribution F . Derive the distribution of the number of cars that are located in the interval (a, b) at time t . (**Hint:** consider to define type-I cars as those which are located in (a, b) .)