

Lecture 3: Markov Chains I (State Classification, Recurrence and Transience)

0.1 Introduction and Examples

- A stochastic process $\{X_n, n = 0, 1, 2, \dots\}$ that takes on a finite or countable number of possible values. We call it a *Markov chain* if the process satisfies that

$$P[X_{n+1} = j | X_0 = i_0, X_1 = i_1, \dots, X_{n-1} = i_{n-1}, X_n = i] = P[X_{n+1} = j | X_n = i]$$

for any states $i_0, i_1, \dots, i_{n-1}, i$ and j and all $n \geq 0$. The equation can be interpreted as stating that, for a Markov chain, the future evolution of the process is independent of its past history if the current state is known. In this course, we study time-homogeneous Markov chains only. In other words, $P[X_{n+1} = j | X_n = i]$ has nothing to do with n . Denote it by p_{ij} . Sometimes people use a *transition probability* matrix

$$\mathbf{P} = \begin{bmatrix} p_{00} & p_{01} & p_{02} & \cdots \\ p_{10} & p_{11} & p_{12} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ p_{i0} & p_{i1} & p_{i2} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

to represent a Markov chain with the state space $\{0, 1, 2, \dots\}$.

- We have

$$p_{ij} \geq 0; \quad \sum_j p_{ij} = 1.$$

Example 1 (Sum of I.I.D. random variables) Let $X_i, i \geq 1$ be i.i.d. random variables with

$$P(X_i = j) = a_j, \quad j = 0, \pm 1, \pm 2, \dots$$

Define

$$S_0 = 0 \text{ and } S_n = \sum_{i=1}^n X_i.$$

Then $\{S_n, n \geq 0\}$ is a Markov chain with $p_{ij} = a_{j-i}$.

Example 2 (The M/G/1 queue) Customers arrive at a service center in accordance with Poisson process with rate λ . There is a single server and those arrivals finding the server free go immediately into service; all others wait in line until their service turn. The service times for customers are assumed to be i.i.d. with a common distribution G . Let $\{X_n, n \geq 0\}$ denote the number of customers left behind in the queue by the n th departure. We can show that $\{X_n, n \geq 0\}$ is a Markov chain with transition probability

$$\begin{aligned} p_{0j} &= \int_0^{+\infty} e^{-\lambda x} \frac{(\lambda x)^j}{j!} dG(x), \quad j \geq 0; \\ p_{ij} &= \int_0^{+\infty} e^{-\lambda x} \frac{(\lambda x)^{j-i+1}}{(j-i+1)!} dG(x), \quad j \geq i-1, \quad i \geq 1, \\ p_{ij} &= 0, \text{ otherwise.} \end{aligned}$$

Example 3 (An Inventory Example) A commodity is stocked in order to satisfy a continuing demand. Replenishing of stock takes place at successive times $t_1, t_2, \dots$. Assume that the cumulative demand for the commodity over the interval (t_{n-1}, t_n) is a random variable ξ_n whose distribution function is

$$P(\xi_n = k) = a_k, \quad k = 1, 2, \dots$$

An inventory policy is prescribed by specifying two nonnegative critical values s and $S > s$. If the available stock quantity is not greater than s then immediate procurement is done so as to bring the stock quantity to S . If, however, the available stock is in excess of s then no replenishment is undertaken. It is easy to see the stock levels at two consecutive periods are connected by the relation

$$X_{n+1} = \begin{cases} X_n - \xi_{n+1}, & \text{if } s < X_n \leq S; \\ S - \xi_{n+1}, & \text{if } X_n \leq s. \end{cases}$$

$\{X_n\}$ is a Markov chain and its transition probabilities can be calculated by this relation.

0.2 Chapman-Kolmogorov Equations

- Define the n -step transition probability p_{ij}^n to be the probability that a process in state i will be in state j after n additional transitions. That is,

$$p_{ij}^n = P(X_{n+m} = j | X_m = i), \quad n \geq 0, \quad i, j \geq 0.$$

- Surely $p_{ij}^1 = p_{ij}$.

Theorem 1 (The Chapman-Kolmogorov Equation)

$$p_{ij}^{n+m} = \sum_k p_{ik}^n p_{kj}^m$$

for all $n, m \geq 0$ and all i, j . The matrix of n -step transition probabilities satisfies

$$\mathbf{P}^{n+m} = \mathbf{P}^n \cdot \mathbf{P}^m.$$

Example 4 (A simple random walk) A drunken man walks along a straight line. He moves one step right with probability p and left with probability $1 - p$. What is the probability that he comes back where he started in n steps?

0.3 Classification of States and Periodicity

- State j is said to be *accessible* from state i if $p_{ij}^n > 0$ for some n . Two states i and j are said to *communicate* with each other if they are accessible to each other. We write $i \leftrightarrow j$.

Theorem 2 Communication satisfies

- $i \leftrightarrow i$;
- if $i \leftrightarrow j$, then $j \leftrightarrow i$;

- if $i \leftrightarrow j$ and $j \leftrightarrow k$, then $i \leftrightarrow k$.

Two states that communicate are said to be in the same *class*. We can break the whole state space into a union of several classes. If there is only one class, i.e., all states communicate with each other, we say that the Markov chain is *irreducible*.

Example 5 There are two classes in the following Markov chain:

$$\mathbf{P}_1 = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 & 0 \\ 1/4 & 3/4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and $\{0\}$, $\{1, 2, \dots, N-1\}$ and $\{N\}$ are three classes in the following Markov chain:

$$\mathbf{P}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ q & 0 & p & 0 & \cdots & 0 & 0 & 0 \\ 0 & q & 0 & p & \cdots & 0 & 0 & 0 \\ \vdots & & & & & & & \\ 0 & 0 & 0 & 0 & \cdots & q & 0 & p \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}.$$

- State i is said to have a *period* of d if d is the greatest common divisor of all integers $n \geq 1$ for which $p_{ii}^n > 0$. That is, if $p_{ii}^n > 0$, then n is divisible by d and d is the greatest integer with such property. A state is *aperiodic* if 1 is the only period.

Example 6 The simple random walk has a period of 2. In a finite Markov chain of N states with transition matrix

$$\mathbf{P}_3 = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

has a period of N .

Theorem 3 If $i \leftrightarrow j$, then $d(i) = d(j)$. In words, periodicity is a property of a class.

0.4 Transience and Recurrence

- For any states i and j define f_{ij}^n to be the probability that, starting in i , the first transition into j occurs at time n . Formally

$$f_{ij}^n = P(X_n = j, X_k \neq j, k = 1, \dots, n-1 | X_0 = i).$$

Let

$$f_{ij} = \sum_{n=1}^{+\infty} f_{ij}^n.$$

f_{ij} is the probability of ever making a transition into state j , given that the process starts in i . We say j to be *recurrent* if $f_{jj} = 1$ and *transient* otherwise.

Theorem 4 *State j is recurrent if and only if*

$$\sum_{n=1}^{+\infty} p_{jj}^n = +\infty.$$

Corollary 1 *In a finite-state Markov chain not all states can be transient.*

Recurrence is also a class property. That is

Corollary 2 *If state i is recurrent and $i \leftrightarrow j$, then j is recurrent and $f_{ji} = 1$.*

Example 7 *For one-dimensional symmetric simple random walk, it is recurrent for all states.*

- 0-1 law. The next theorem shows that if recurrence is certain for a specified state, then the state will be visited infinitely often with probability 1. On the contrary, the transient state will be visited only finite times. Define

$$q_{ij} = P(\text{a process starting in state } i \text{ visits infinitely often to state } j).$$

Theorem 5 *State i is recurrent or transient according to whether $q_{ii} = 1$ or $q_{ii} = 0$, respectively.*

Corollary 3 *If state i is recurrent and $i \leftrightarrow j$, then $q_{ij} = 1$.*

- Closeness of recurrent classes. A recurrent class is *closed* in the sense that once entered it is never left.

Theorem 6 *Let R be a recurrent class of states. If $i \in R$ and $j \notin R$, then $p_{ij} = 0$.*

Let j be a given recurrent state and T be the set of all transient states. For $i \in T$, we are often interested in computing f_{ij} . The following theorem yields a set of equations satisfied by f_{ij} .

Theorem 7 *If j is recurrent, then the set of probabilities $\{f_{ij}, i \in T\}$ satisfies*

$$f_{ij} = \sum_{k \in T} p_{ik} f_{kj} + \sum_{k \in R} p_{ik}, \quad i \in T.$$

0.5 Some Applications

- **The gambler's ruin problem.** Consider a gambler who at each play of the game has a probability p winning \$1 and probability $1 - p$ losing \$1. Assume that successive plays of the game are independent, what is the probability that, starting with i dollars, the gambler will reach \$N before bankrupt? (Textbook pp. 186 - 187) What is the expected number of bets that the gambler, starting at i , makes before reaching either 0 or N ? (Textbook p. 188)

- **Branching processes.** Consider a population consisting of individuals able to produce offspring of the same kind. Suppose that each individual will have produced j new offsprings with probability $p_j, j \geq 0$, by the end of its lifetime, independently of the number produced by any other individuals. The number of individuals initially present, denoted by X_0 , is called the size of the zeroth generation. All offsprings of the zeroth generation constitute the first generation and their number is denoted by X_1 . Let X_n denote the size of the n th generation. The Markov chain $\{X_n, n \geq 0\}$ is called a *branching process*. What is the probability that the population dies out? (Textbook pp. 191 - 193)

Homework (Due on Sept. 28):

1. Prove that

$$p_{ij}^n = \sum_{k=0}^n f_{ij}^k p_{jj}^{n-k}.$$

2. For state $i, j, k, k \neq j$, let

$$p_{ij/k}^n = P(X_n = j, X_l \neq k, l = 1, 2, \dots, n-1 | X_0 = i).$$

- (a). Explain in words what $p_{ij/k}^n$ represents.
- (b). Prove that, for $i \neq j$,

$$p_{ij}^n = \sum_{k=0}^n p_{ii}^k p_{ij/i}^{n-k}.$$

3. Show that the symmetric random walk is transient in the three dimensional integer grid.
Hint: Use the Stirling approximation

$$n! \sim \sqrt{2\pi n} n^{n+1/2} e^{-n}.$$

4. Let $X_1, X_2, \dots$ be independent random variables such that $P(X_i = j) = a_j, j \geq 0$. Say that a record occurs at time n if $X_n > \max(X_1, \dots, X_{n-1})$ where $X_0 = -\infty$, and if a record does occur at time n call X_n the record value. Let R_i denote the i th record value.
 - (a). Argue that $\{R_i, i \geq 1\}$ is a Markov chain and compute its transition probabilities.
 - (b). Let T_i be the time between the i th and $(i+1)$ st record. Is $\{T_i, i \geq 1\}$ a Markov chain? What about $\{(R_i, T_i), i \geq 1\}$? Compute transition probabilities where appropriate.
 - (c). Now suppose that X_i 's are all continuous random variables. Let $S_n = \sum_{i=1}^n T_i, n \geq 1$. Argue that $\{S_n, n \geq 1\}$ is a Markov chain and find its transition probabilities.

5. For a Markov chain $\{X_n, n \geq 0\}$, show that

$$P(X_k = i_k | X_j = i_j \text{ for all } j \neq k) = P(X_k = i_k | X_{k-1} = i_{k-1}, X_{k+1} = i_{k+1}).$$

6. In the gambler's ruin problem show that

$$\begin{aligned} & P(\text{she wins the next gamble} | \text{present fortune is } i, \text{ she eventually reaches } N) \\ &= \begin{cases} p[1 - (q/p)^{i+1}] / [1 - (q/p)^i], & \text{if } p \neq 1/2; \\ (i+1)/2i & \text{if } p = 1/2. \end{cases} \end{aligned}$$