

Lecture 3: Markov Chains II (Limit Theorems and Stationary Distributions)

0.1 Limit Theorems

- A key tool in the analysis of Markov chains is furnished by the following theorem. One may refer to Karlin and Taylor, the second reference book, Chapter 3 for the proof.

Theorem 1 (Discrete Renewal Theorem) *Let $\{a_k\}, \{u_k\}, \{b_k\}$ be sequences indexed by $k = 0, 1, 2, \dots$. Suppose that $a_k \geq 0$, $\sum_k a_k = 1$, $\sum_k |k|a_k < +\infty$, $\sum_k |b_k| < +\infty$, and that the greatest common divisor of the integers k for which $a_k > 0$ is 1. If the equation*

$$u_n - \sum_{k=0}^n a_{n-k} u_k = b_n$$

for $n = 0, 1, 2, \dots$ is satisfied by a bounded sequence $\{u_n\}$ of real numbers, then $\lim_{n \rightarrow +\infty} u_n$ exists and

$$\lim_{n \rightarrow +\infty} u_n = \frac{\sum_{k=0}^{\infty} b_k}{\sum_{k=0}^{\infty} k a_k}.$$

In case $\sum_{k=-\infty}^{\infty} k a_k = +\infty$, the limit relations are still valid provided we interpret the fraction to be 0.

- Let μ_j be the expected number of transitions needed to return to state j . That is,

$$\mu_j = \begin{cases} +\infty, & \text{if } j \text{ is transient;} \\ \sum_{n=1}^{+\infty} n f_{jj}^n, & \text{if } j \text{ is recurrent.} \end{cases}$$

We have the following major theorem about the limits of Markov chains:

Theorem 2 (i). *Consider a recurrent irreducible aperiodic Markov chain. Then,*

$$\lim_{n \rightarrow +\infty} p_{ij}^n = \lim_{n \rightarrow +\infty} p_{jj}^n = 1/\mu_j.$$

(ii). *If j is a member of a recurrent periodic class with period d , then*

$$\lim_{n \rightarrow +\infty} p_{jj}^{nd} = d/\mu_j.$$

(iii). *If j is transient, $\lim_{n \rightarrow +\infty} p_{jj}^n = 0$.*

Corollary 1

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{k=1}^n p_{ij}^k = \frac{1}{\mu_j}.$$

0.2 Positive Recurrence, Null Recurrence and Stationary Distribution

- If $\lim_{n \rightarrow +\infty} p_{jj}^n > 0$, we say state j is *positive recurrent* or *strongly ergodic*. For a recurrent state j , if $\lim_{n \rightarrow +\infty} p_{jj}^n = 0$, j is called *null recurrent* or *weakly ergodic*.

Theorem 3 *Positive (null) recurrence is a class property.*

- Consider a positive recurrent aperiodic irreducible Markov chain. There exists a nonnegative vector $\pi = \{\pi_i\}$ such that $\sum_i \pi_i = 1$ and

$$\pi_j = \sum_i \pi_i p_{ij}.$$

In words, if the initial position of the Markov chain is chosen according to a distribution π , then the distribution of the state of the Markov chain will remain unchanged. We call such distribution by *stationary*.

Theorem 4 *An irreducible aperiodic Markov chain belongs to one of the following two classes:*
 (i). *Either the states are all transient or all null recurrent; in this case, $p_{ij}^n \rightarrow 0$ as $n \rightarrow +\infty$ for all i and j and there exists no stationary distribution.*
 (ii). *Or else, all states are positive recurrent; in this case, $\{\pi_i := \lim_{n \rightarrow +\infty} p_{ii}^n\}$ is a stationary distribution and there exists no other stationary distribution. We say that the Markov chain is ergodic.*

From the stationary distribution, we may proceed to obtain the following ergodic theorem

Theorem 5 *Suppose that $\{X_n, n \geq 0\}$ is an irreducible positive recurrent Markov chain. g is a bounded function. Then,*

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=1}^n g(X_i) = \sum_j \pi_j g(j).$$

- The calculation of π can be done by solving the following linear equations:

$$\pi \cdot \mathbf{P} = \mathbf{P}, \quad \sum_i \pi_i = 1.$$

Example 1 *Limiting probability of M/G/1 queue.* Consider the Markov chain embedded in the M/G/1 queue in the last lecture. Let

$$a_j = \int_0^{+\infty} e^{-\lambda t} \frac{(\lambda t)^j}{j!} dG(t).$$

Then the transition probabilities for this chain are

$$\begin{aligned} p_{0i} &= a_i, & p_{ij} &= a_{j-i+1}, \quad i > 0, j \geq i-1, \\ p_{ij} &= 0, & j &< i-1. \end{aligned}$$

The stationary distribution satisfies

$$\pi_j = \pi_0 a_j + \sum_{i=1}^{j+1} \pi_i a_{j-i+1}.$$

We can solve for π by considering its generating function.

Example 2 The age of a renewal process. Initially a light bulb is put into use, and when it fails it is replaced at the beginning of the next time period by a new bulb. Suppose that the lives of all bulbs are independent and each will fail in its i th period of use with probability q_i , $i \geq 1$ and $\sum_i i p_i < +\infty$. Let X_n denote the age of the bulb in use at time n — that is, the number of periods it has been in use. $\{X_n\}$ is a Markov chain with transition probabilities given by

$$p_{i1} = \frac{q_i}{\sum_{j=i}^{+\infty} q_j} = 1 - p_{i,i+1}.$$

The limiting probabilities are such that

$$\begin{aligned}\pi_1 &= \sum_i \pi_i \lambda_i, \\ \pi_{i+1} &= \pi_i (1 - \lambda_i), \quad i \geq 1.\end{aligned}$$

0.3 Absorption Probability and Expected Number of Time Periods Spent in a Transient State

- For transient states i and j , let m_{ij} denote the expected total number of time periods spent in j given that the chain starts in state i . Conditioning on the initial transition yields

$$m_{ij} = \begin{cases} 1 + \sum_{k=1}^t p_{ik} m_{kj}, & j = i; \\ \sum_{k=1}^t p_{ik} m_{kj}, & j \neq i. \end{cases} \quad (1)$$

Let

$$\mathbf{M} = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1t} \\ m_{21} & m_{22} & \cdots & m_{2t} \\ \vdots & \vdots & \ddots & \vdots \\ m_{t1} & m_{t2} & \cdots & m_{tt} \end{bmatrix}.$$

In matrix notation, (1) can be written as

$$\mathbf{M} = \mathbf{I} + \mathbf{Q}\mathbf{M}$$

where $\mathbf{I}$ is the identity matrix of size t . We have

$$\mathbf{M} = (\mathbf{I} - \mathbf{Q})^{-1}.$$

- For $i, j \in T$, the quantity f_{ij} , equal to the probability of ever making a transition into state j given that the chain starts in state i , can be determined from $\mathbf{M}$ by the relationship

$$m_{ij} = m_{jj} f_{ij}.$$

Example 3 The gambler's ruin problem (continued). Consider the gambler ruin problem with $p = 0.4$ and $N = 6$. Starting with \$3, determine:

- the expected amount of time spent in state 3.
- the expected number of visits to state 2.
- the probability of ever visiting state 4.

Homework:

The homework questions may use some facts from the last lecture.

1. If $f_{ii} < 1$ and $f_{jj} < 1$, show that
 - (a). $\sum_{n=1}^{+\infty} p_{ij}^n < +\infty$.
 - (b). $f_{ij} = \frac{\sum_{n=1}^{+\infty} p_{ij}^n}{1 + \sum_{n=1}^{+\infty} p_{jj}^n}$.
2. Compute the expected number of plays, starting in i , in the gambler's ruin problem, until the gambler reaches either 0 or N .
(Hint: Consider $E[\sum_{i=1}^{\tau} X_i]$, where τ is the first time the gambler reaches 0 or N and X_i is the gain/loss of the i th play. Make use of the Wald equation to obtain one expression for the expectation. Use the probabilities reaching 0 or N to obtain another. Do a comparison to solve the problem.)
3. Consider a Markov chain with states $0, 1, \dots, n$ and transition probabilities

$$p_{0,1} = 1 = p_{n,n-1}, \quad p_{i,i+1} = p = 1 - p_{i,i-1}, \quad 0 < i < n.$$

Let $\mu_{i,n}$ denote the mean time to go from state i to state n .

- (a). Derive a set of linear equations for the $\mu_{i,n}$.

(Hint: Conditioning on the first move of the process, you can express $\mu_{i,n}$ in terms of $\mu_{i+1,n}$ and $\mu_{i-1,n}$)

- (b). Let m_i be the mean time to go from state i to state $i+1$. Derive a set of equations for the m_i , $i = 0, \dots, n-1$, and show how they can be solved recursively, first for $i = 0$, then $i = 1$, and so on.

- (c). What is the relation between $\mu_{i,n}$ and m_j ?

- (d). Starting at state 0, say that an excursion ends when the chain either returns to 0 or reaches state n . Let X_j denote the number of transitions in the j th excursion (that is, the one that begins at the j th return to 0), $j \geq 1$. Find $E[X_j]$.

(Hint: Relate it to the mean time of a gambler's ruin problem)

- (e). Let N be the rounds of excursions until the first excursion ends in state n , and find $E[N]$.
- (f). Find $\mu_{0,n}$ and $\mu_{i,n}$.

4. A spider hunting a fly moves between locations 1 and 2 according to a Markov chain with transition matrix

$$\begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix},$$

starting in location 1. The fly, unaware of the spider, starts in location 2 and moves according to another Markov chain with transition matrix

$$\begin{bmatrix} 0.4 & 0.6 \\ 0.6 & 0.4 \end{bmatrix}.$$

The spider catches the fly and the hunt ends whenever they meet in the same location.

- (a). Show that the progress of the hunt, except for knowing the location where it ends, can

be described by a three-state Markov chain where one absorbing state represents hunt ended and the other two that the spider and fly are at different locations. Obtain the transition matrix for this chain.

(b). Find the probability that at time n the spider and fly are both at their initial locations.

(c). What is the average duration of the hunt?

5. Consider a branching process $\{X_n, n \geq 0\}$. Show that

$$\text{Var}(X_n | X_0 = 1) = \begin{cases} \sigma^2 \mu^{n-1} \frac{\mu^n - 1}{\mu - 1}, & \text{if } \mu \neq 1 \\ n\sigma^2, & \text{if } \mu = 1, \end{cases}$$

where μ and σ^2 are the means and variance of the number of offsprings an individual has.