

Lecture 3: Markov Chains III (Time Reversibility)

0.1 Time-Reversible Markov Chains

- Consider a stationary Markov chain having transition probabilities p_{ij} and stationary distribution π_i . Trace the sequence of states going backwards in time. This sequence of state itself is a Markov chain with transition probability p_{ij}^* defined by

$$p_{ij}^* = P(X_m = j | X_{m+1} = i) = \frac{P(X_{m+1} = i | X_m = j)P(X_m = j)}{P(X_{m+1} = i)} = \frac{\pi_j p_{ji}}{\pi_i}.$$

- If $p_{ij}^* = p_{ij}$ for all i, j , then the Markov chain is said to be *time reversible*. It satisfies

$$\pi_i p_{ij} = \pi_j p_{ji} \quad (1)$$

for all i, j . Sometimes, people refer to this equality as *detailed balance condition*. It can be interpreted as stating that, the “rate” at which the process goes from i to j is equal to the “rate” at which it goes from j to i .

If we can find non-negative numbers, summing to 1, which satisfy (1), then it follows that the Markov chain is time reversible and the numbers represent the stationary probabilities. This observation provides us a simple approach to find the stationary probabilities of a Markov chain.

Example 1 Consider the Markov chain with states $0, 1, \dots, n$ and with transition probabilities

$$\begin{aligned} p_{0,1} &= p_{n,n-1} = 1; \\ p_{i,i+1} &= p_i = 1 - p_{i,i-1}, \quad i = 1, \dots, n-1. \end{aligned}$$

Find its stationary probabilities.

We can extend the above conclusion to the following theorem:

Theorem 1 Consider an irreducible Markov chain with transition probabilities p_{ij} . If one can find nonnegative numbers π_i , summing to unity, and a transition probability matrix $\mathbf{P}^* = [p_{ij}^*]$ such that

$$\pi_i p_{ij} = \pi_j p_{ji}^*,$$

then the π_i are the stationary probabilities and p_{ij}^* are the transition probabilities of the reversed chain.

- Time-reversible Markov chain has wide applications in OR. One example is taken from simulation area.

Example 2 (The Metropolis-Hasting Algorithm) Let $a_j, j = 1, \dots, m$ be positive numbers, and let $A = \sum_{j=1}^m a_j$. Suppose that m is large and that A is difficult to compute. Ideally,

we want to simulate the values of a sequence of independent random variables whose probabilities are $p_j = a_j/A$. One way of simulating a sequence of random variables is to find a Markov chain that is easy to simulate and whose limiting probabilities are the p_j . The MH algorithm provides an approach for accomplishing this task.

Let $\mathbf{Q}$ be any irreducible probability matrix on the integers $1, \dots, n$ such that $q_{ij} = q_{ji}$. Define a Markov chain $\{X_n, n \geq 0\}$ as follows. If $X_n = i$, then generate a random variable that is equal to j with probability q_{ij} . If this random variable takes on the value j , then set $X_{n+1} = j$ with probability $\min\{1, a_j/a_i\}$ and set it equal to i otherwise. The limiting probabilities of this Markov chain are precisely the p_j .

- If we try to solve the detailed balance condition (1) for an arbitrary Markov chain, it will usually turn out that no solution exist.

Theorem 2 A stationary Markov chain is time reversible if and only if, starting in state i , any path back to i has the same probability as the reversed path for all i . That is, if

$$p_{ii_1} \cdot p_{i_1 i_2} \cdot \dots \cdot p_{i_k i} = p_{ii_k} \cdot p_{i_k i_{k-1}} \cdot \dots \cdot p_{i_1 i}$$

for all states $i_1, \dots, i_k$.

Homework:

The homework questions may use some facts from the last lecture.

1. M balls are initially distributed among m urns. At each stage one of the balls are selected at random, taken from whichever urn it is in, and placed, at random, in one of the other $m - 1$ urns. Consider the Markov chain whose state at any time is the vector $(n_1, \dots, n_m)$, where n_i denotes the number of balls in urn i . Guess at the limiting probabilities for this Markov chain and then verify your guess and show at the same time that the Markov chain is time reversible.
2. Show that a finite state ergodic Markov chain such that $p_{ij} > 0$ for all $i \neq j$ is time reversible if and only if

$$p_{ij}p_{jk}p_{ki} = p_{ik}p_{kj}p_{ji}$$

for all i, j, k .

3. Consider a time-reversible Markov chain with transition probabilities p_{ij} and limiting probabilities π_i ; and now consider the same chain truncated to the states $0, 1, \dots, M$. That is, for the truncated chain its transition probabilities $\bar{p}_{ij}$

$$\bar{p}_{ij} = \begin{cases} p_{ij} + \sum_{k>M} p_{ik}, & 0 \leq i \leq M, j = i; \\ p_{ij}, & 0 \leq i \neq j \leq M; \\ 0, & \text{Otherwise.} \end{cases}$$

Show that the truncated chain is also time reversible and has limiting probabilities given by

$$\bar{\pi}_i = \frac{\pi_i}{\sum_{i=0}^M \pi_i}.$$