

Lecture 4: Continuous Time Markov Chain I

0.1 Continuous Time Markov Chains and Kolmogorov Differential Equations

- $\{X_t, t \geq 0\}$ is a *continuous time Markov chain* if for all $s, t \geq 0$, $i, j \in \mathcal{S}$, $x_u \in \mathcal{S}$, $0 \leq u \leq s$,

$$P[X_{t+s} = j | X_s = i, X_u = x_u, 0 \leq u \leq s] = P[X_{t+s} = j | X_s = i].$$

All Markov chains we consider will be assumed to have time homogeneous transition probabilities. That is, $P[X_{t+s} = j | X_s = i]$ is independent of s . A typical example of continuous-time Markov chain is Poisson process. For all $i, j \in \mathcal{S}$. Let

$$p_{ij}(t) = P[X_{t+s} = j | X_s = i].$$

We have the continuous-time version of Chapman-Kolmogorov equation such that

$$p_{ij}(t+s) = \sum_{k \in \mathcal{S}} p_{ik}(t) p_{kj}(s).$$

- Suppose that a Markov chain starts from i and $\tau_i = \inf\{t \geq 0 : X_t \neq i\}$. We have

$$P[\tau_i > t+s] = P[\tau_i > t] \cdot P[\tau_i > s]$$

for all $t, s \geq 0$, which implies that there must exist a number v_i such that $P[\tau_i > t] = \exp(-v_i t)$. Here v_i satisfies $0 \leq v_i \leq +\infty$. When $v_i = 0$, the chain will not leave state i . We call such state *absorbing*. When $v_i = +\infty$, the chain will instantaneously leave it when it enters state i . We call such state *instantaneous*. Let

$$p_{ij} = P[X_{\tau_i} = j | X_0 = i].$$

Then $\sum_{j \in \mathcal{S}} p_{ij} = 1$. $[p_{ij}]$ constitutes a transition probability matrix. We call the discrete-time Markov chain corresponding to $[p_{ij}]$ as the *embedded chain* of the original continuous-time Markov chain. Given $\{v_i\}$ and $[p_{ij}]$, we then can construct a continuous-time Markov chain in two steps:

1. The amount of time the Markov chain spends in state i before making a transition is exponentially distributed with rate v_i ;
2. When the process leaves i , it will next enter j with probability p_{ij} .

Define $q_{ij} = v_i p_{ij}$.

- **Theorem 1 (Kolmogorov Backward-Forward Equations)** (i). $v_i = \lim_{t \rightarrow 0} \frac{1-p_{ii}(t)}{t}$ and

$$q_{ij} = \lim_{t \rightarrow 0} \frac{p_{ij}(t)}{t}.$$

(ii) For all i, j and $t \geq 0$,

$$p'_{ij}(t) = \sum_{k \neq i} q_{ik} p_{kj}(t) - v_i p_{ij}(t).$$

(iii) For all i, j and $t \geq 0$,

$$p'_{ij}(t) = \sum_{k \neq j} q_{kj} p_{ik}(t) - v_j p_{ij}(t).$$

Example 1 (Two-State Chain) Consider a two-state continuous-time chain that spends an exponential time with rate λ in state 0 before going to state 1, where it spends an exponential time with rate μ before returning to state 0. We have a forward equation

$$p'_{00}(t) = \mu p_{00}(t) - \lambda p_{00}(t) = -(\lambda + \mu)p_{00}(t) + \mu.$$

Solve this ordinary differential equation. Then,

$$p_{00}(t) = \frac{\mu}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} e^{-(\lambda + \mu)t}.$$

- Define

$$r_{ij} = \begin{cases} q_{ij}, & \text{if } i \neq j; \\ -v_i, & \text{if } i = j. \end{cases}$$

Let $\mathbf{Q} = (r_{ij})$. According to Kolmogorov backward equation, we have

$$\mathbf{P}'(\mathbf{t}) = \mathbf{Q}\mathbf{P}(\mathbf{t});$$

and the forward equation can be represented as

$$\mathbf{P}'(\mathbf{t}) = \mathbf{P}(\mathbf{t})\mathbf{Q}.$$

This suggests the solution

$$\mathbf{P}(t) = e^{\mathbf{R}t} = \sum_{i=0}^{+\infty} (\mathbf{R}t)^i / i!.$$

0.2 Birth and Death Processes

- A continuous-time Markov chain with states $0, 1, \dots$ for which $q_{ij} = 0$ whenever $|i - j| > 1$ is called a *birth-death process*. Let

$$\lambda_i = q_{i,i+1}, \quad \mu_i = q_{i+1,i}$$

Since $\sum_j q_{ij} = v_i$, we see that $v_i = \lambda_i + \mu_i$ and

$$p_{i,i+1} = \frac{\lambda_i}{\lambda_i + \mu_i}.$$

Example 2 (M/M/s queue) Suppose that customers arrive at an s -server station in accordance with a Poisson process having rate λ . The successive service times are assumed to be independent exponential random variables having mean $1/\mu$. Let $X(t)$ denote the number in the system at time t , then it is a birth and death process with $\lambda_n = \lambda$ and

$$\mu_n = \begin{cases} n\mu, & \text{if } 1 \leq n \leq s; \\ s\mu, & \text{if } n > s. \end{cases}$$

Example 3 (Linear Growth Model with Immigration) *A model in which*

$$\lambda_n = n\lambda + \theta, \quad \mu_n = n\mu$$

is called a linear growth process with immigration. The interpretation of such model is that each individual in a population gives a birth at an exponential rate λ and there is an exponential rate of increase θ of the population due to an external source such as immigration. Deaths are assumed to occur at an exponential rate μ for each member of the population.

- A birth and death process is said to be a *pure birth* process if $\mu_n = 0$ for all n . In the linear growth model with immigration, if $\mu_n = 0$ and $\theta = 0$, we call it a *Yule process*. Consider a Yule process starting with a single individual at time 0, and let T_i be the time between $(i - 1)$ st and i th birth. We can see

$$\begin{aligned} P[T_1 \leq t] &= 1 - e^{-\lambda t}; \\ P[T_1 + T_2 \leq t] &= \int_0^t P[T_1 + T_2 \leq t | T_1 = x] \lambda e^{-\lambda x} dx = (1 - e^{-\lambda t})^2; \\ P[T_1 + T_2 + T_3 \leq t] &= \int_0^t P[T_1 + T_2 + T_3 \leq t | T_1 + T_2 = x] 2\lambda e^{-\lambda x} (1 - e^{-\lambda x}) dx \\ &= (1 - e^{-\lambda t})^3 \end{aligned}$$

and in general,

$$P[T_1 + T_2 + \cdots + T_j \leq t] = (1 - e^{-\lambda t})^j.$$

Hence, for a Yule process,

$$\begin{aligned} p_{1j}(t) &= P[X(t) = j | X_0 = 1] = P[X(t) \geq j | X_0 = 1] - P[X(t) \geq j + 1 | X_0 = 1] \\ &= P[T_1 + T_2 + \cdots + T_j \leq t | X_0 = 1] - P[T_1 + T_2 + \cdots + T_j + T_{j+1} \leq t | X_0 = 1] \\ &= e^{-\lambda t} (1 - e^{-\lambda t})^{j-1}. \end{aligned}$$

For a population with i members initially, it may be decomposed into i sub-populations with one ancestor and thus we can see

$$p_{ij}(t) = \binom{j-1}{i-1} e^{-\lambda t i} (1 - e^{-\lambda t})^{j-i}.$$

Another interesting result about the Yule process concerns about the conditional distribution of the times of birth given the population size at t . Let $S_i = T_1 + \cdots + T_i$.

$$\begin{aligned} &P[S_1 = s_1, S_2 = s_2, \dots, S_n = s_n | X(t) = n + 1] \\ &= \frac{P[T_1 = s_1, T_2 = s_2 - s_1, \dots, T_n = s_n - s_{n-1}, T_{n+1} > t - s_n]}{P[X(t) = n + 1]} \\ &= \frac{\lambda e^{-\lambda s_1} \cdot 2\lambda e^{-2\lambda(s_2 - s_1)} \cdots n\lambda e^{-n\lambda(s_n - s_{n-1})} e^{-(n+1)\lambda(t - s_n)}}{P[X(t) = n + 1]} \\ &= n! \cdot \frac{e^{-\lambda(t - s_1)}}{1 - e^{-\lambda t}} \cdot \frac{e^{-\lambda(t - s_2)}}{1 - e^{-\lambda t}} \cdots \frac{e^{-\lambda(t - s_n)}}{1 - e^{-\lambda t}}. \end{aligned}$$

Therefore, the birth times are distributed as the ordered values from a sample of size n from a random variable having density

$$f(x) = \begin{cases} \frac{\lambda e^{-\lambda(t-x)}}{1-e^{-\lambda t}}, & \text{if } 0 \leq x \leq t; \\ 0, & \text{otherwise.} \end{cases}$$

0.3 Limiting Probabilities

- Study the limiting property of a continuous-time Markov chain. Start with the forward equation,

$$p'_{ij}(t) = \sum_{k \neq j} q_{kj} p_{ik}(t) - v_j p_{ij}(t).$$

If we assume that the limiting probabilities $p_j = \lim_{t \rightarrow +\infty} p_{ij}(t)$ exist, then $\lim_{t \rightarrow +\infty} p'_{ij}(t)$ would necessarily be 0. Hence, assuming that we can interchange limit and summation in the above, we obtain upon letting $t \rightarrow +\infty$,

$$0 = \sum_{k \neq j} q_{kj} p_k - v_j p_j.$$

That is,

$$v_j p_j = \sum_{k \neq j} q_{kj} p_k \text{ for all } j. \quad (1)$$

- Suppose that the embedded discrete Markov chain of the above continuous-time process has transition probabilities $[p_{ij}]$ and it is irreducible and positive recurrent. Then, according to the discussion in the last lecture, there should exist a unique nonnegative solution $\{\pi_i\}$ to the following linear system:

$$\sum_i \pi_i = 1; \quad \pi_j = \sum_k p_{kj} \pi_k \text{ for all } j.$$

When $\sum_j \pi_j / v_j < +\infty$, let

$$p_j = \frac{\pi_j / v_j}{\sum_j \pi_j / v_j}.$$

Such $\{p_j\}$ must be a unique nonnegative solution to the linear system (1) and $\sum_j p_j = 1$. From this, we claim

Theorem 2 *If the embedded chain of a continuous-time Markov process is irreducible and positive recurrent, then the limiting probabilities $p_j = \lim_{t \rightarrow +\infty} p_{ij}(t)$ are given by*

$$p_j = \frac{\pi_j / v_j}{\sum_j \pi_j / v_j}.$$

In addition, p_j also equals the long run proportion of time the process is in state j , i.e.,

$$p_j = \lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t \mathbf{1}_{\{X_u = j\}} du.$$

A more rigorous proof needs renewal theory. We put it off to later lectures.

- If the initial state is chosen according to the limiting probabilities $\{p_j\}$, then the resulting process will be stationary. That is,

$$\sum_i p_i p_{ij}(t) = p_j$$

for all t .

0.4 Limiting Probabilities of Birth-Death Processes

- Consider a birth-death process on the real line. From (1), we obtain

$$\begin{aligned}\lambda_0 p_0 &= \mu_1 p_1; \\ (\lambda_n + \mu_n) p_n &= \mu_{n+1} p_{n+1} + \lambda_{n-1} p_{n-1} \text{ for all } n > 0.\end{aligned}$$

Rewriting these equations gives

$$\begin{aligned}\lambda_0 p_0 &= \mu_1 p_1; \\ \lambda_n p_n &= \mu_{n+1} p_{n+1} + (\lambda_{n-1} p_{n-1} - \mu_n p_n) \text{ for all } n > 0.\end{aligned}$$

or equivalently,

$$\begin{aligned}\lambda_0 p_0 &= \mu_1 p_1; \\ \lambda_1 p_1 &= \mu_2 p_2 + (\lambda_0 p_0 - \mu_1 p_1) = \mu_2 p_2; \\ \lambda_2 p_2 &= \mu_3 p_3 + (\lambda_1 p_1 - \mu_2 p_2) = \mu_3 p_3; \\ &\dots\dots \\ \lambda_n p_n &= \mu_{n+1} p_{n+1} + (\lambda_{n-1} p_{n-1} - \mu_n p_n) = \mu_{n+1} p_{n+1}.\end{aligned}$$

Solving in terms of p_0 yields

$$\begin{aligned}p_1 &= \frac{\lambda_0}{\mu_1} p_0; \\ p_2 &= \frac{\lambda_1}{\mu_2} p_1 = \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} p_0; \\ &\dots\dots \\ p_n &= \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} p_0.\end{aligned}$$

Using $\sum_{n=0}^{+\infty} p_n = 1$, we have

$$p_0 = \left[1 + \sum_{n=1}^{+\infty} \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} \right]^{-1}$$

and hence other p_n 's. One condition is needed for the limiting probabilities to exist, which is

$$\sum_{n=1}^{+\infty} \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} < +\infty.$$

Example 4 In $M/M/1$ queue, $\lambda_n = \lambda$, $\mu_n = \mu$ and thus the limiting probability

$$p_n = \frac{(\lambda/\mu)^n}{1 + \sum_{n=1}^{+\infty} (\lambda/\mu)^n} = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right).$$

Homework (Due on Nov. 5):

1. For a pure birth process with birth parameters $\lambda_n, n \geq 0$, compute the moment generating function of the time it takes the population to go from 0 to N . And then calculate its mean and variance.

(**Hint:** Let T_i be the time the process takes from i to $i + 1$. Then you need to calculate $E[e^{-s(T_0 + \dots + T_{N-1})}]$).

2. Suppose that a machine has two states $\{0, 1\}$. When it is in state $i, i = 0, 1$, it will produce defect items in accordance with a Poisson process with rate α_i . Suppose that the machine's state can be modeled as a two-state continuous-time Markov chain with transition rates $v_0 = \lambda$ and $v_1 = \mu$. Let $N(t)$ denote the total number of defect items in $(0, t)$.

(a). What is

$$\lim_{h \rightarrow 0} \frac{E[N(t+h) - N(t)]}{h}?$$

(**Hint:** Conditioning on the state of the machine at time t , you may be able to calculate $E[N(t+h) - N(t)]$ using the property of Poisson process).

(b). If the initial state is 0, find $E[N(t)]$. (**Hint:** Try to use part (a) to establish an ordinary differential equation about $E[N(t)]$.)

3. What is the stationary distribution of the linear growth birth and death process when $\lambda < \mu$ and $\theta = 0$?