

Lecture 4: Continuous Time Markov Chain II

0.1 Time Reversibility

- Consider an ergodic Markov chain which has been in operation an infinitely long time. We can assume that it started at time $-\infty$. Such process will be stationary. Let us consider this process going backward in time. Like the discrete-time Markov chain, the reversed process is also a continuous-time Markov chain because

$$P[X(t-s) = j | X(t) = i, X(y), y > t] = P[X(t-s) = j | X(t) = i].$$

- For the reversed process, the amount of time that the process spends in state i during a visit is also exponentially distributed with the same rate v_i as the forward-going process. In fact,

$$P[\text{process is in state } i \text{ throughout } [t-s, t] | X_t = i] = \frac{P[X(t-s) = i]e^{-v_i s}}{P[X(t) = i]} = e^{-v_i s}.$$

Meanwhile, the embedded discrete-time chain should be with transition probabilities

$$p_{ij}^* = \frac{\pi_j p_{ji}}{\pi_i}$$

where $\{\pi_i\}$ are the stationary distributions of the embedded chain of the original Markov chain. Let q_{ij}^* be the infinitesimal rates of the reversed process.

$$q_{ij}^* = v_i p_{ij}^* = \frac{v_i \pi_j p_{ji}}{\pi_i}.$$

In addition, recalling that

$$p_j = \frac{\pi_j / v_j}{\sum_j \pi_j / v_j},$$

we see

$$\frac{\pi_j}{\pi_i} = \frac{v_j p_j}{v_i p_i}$$

and so,

$$q_{ij}^* = \frac{v_j p_j p_{ji}}{p_i} = \frac{p_j q_{ji}}{p_i}.$$

That is,

$$p_i q_{ij}^* = p_j q_{ji}. \quad (1)$$

From (1), we can argue that $\{p_i\}$ must be the stationary distribution of the reversed chain. In fact, summing (1) over all states i ,

$$\sum_i p_i q_{ij}^* = \sum_i p_j q_{ji} = p_j v_j.$$

Since q_{ij}^* is the rate, (1) has a very nice and intuitive interpretation. The rate at which the reversed chain makes a transition from i to j equals to the rate at which the forward-going chain makes a transition from j to i .

- Consider a continuous-time Markov chain whose state space is $\mathcal{S}$. We say that the Markov chain is *truncated* to the set $A \subset \mathcal{S}$ if q_{ij} is changed to 0 for all $i \in A$ and $j \notin A$ and all other q_{ij} remain the same. Thus transitions out of the class of states A are not allowed.

Theorem 1 *A time-reversible chain with limiting probabilities $p_j, j \in S$, that is truncated to the set $A \subset S$ and remains irreducible is also time reversible and has limiting probabilities*

$$p_j^A = \frac{p_j}{\sum_{j \in A} p_j}.$$

Example 1 (M/M/1/N queue) *Consider an M/M/1 queue in which arrivals finding N in system do not enter but rather are lost. The limiting probability is given by*

$$p_j = \frac{(\lambda/\mu)^j}{\sum_{n=0}^N (\lambda/\mu)^n}.$$

0.2 Uniformization

- Consider a Markov chain for which the exit rates of all states, $\{v_i\}$, are bounded. Let v be any upper bound. We may construct such Markov chain in the following way: the process spends an exponential amount of time with rate v in state i and then makes a transition to j with probability p_{ij}^* such that

$$p_{ij}^* = \begin{cases} 1 - \frac{v_i}{v}, & j = i; \\ \frac{v_i}{v} p_{ij}, & j \neq i. \end{cases}$$

Hence, the transition probabilities can be computed by

$$p_{ij}(t) = \sum_{n=0}^{+\infty} p_{ij}^{*n} e^{-vt} \frac{(vt)^n}{n!}.$$

This technique of uniformizing the rate in which a transition occurs from each state by introducing transitions from a state to itself is known as *uniformization*.

Example 2 (Two-state chain) *Consider the two-state chain again. Let $v = \lambda + \mu$. We have*

$$p_{00}(t) = \sum_{n=0}^{+\infty} p_{00}^{*n} e^{-(\lambda+\mu)t} \frac{((\lambda+\mu)t)^n}{n!} = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}.$$

Homework (No Submission is Required):

1. Each individual in a biological population is assumed to give birth at an exponential rate λ and to die at exponential rate μ , In addition, there is an exponential rate of increase θ due to immigration. However, immigration is not allowed when the population size is N or larger.
 - (a). Set this up as a birth-death model.
 - (b). If $N = 3$, $\theta = \lambda = 1$ and $\mu = 2$, determine the proportion of time that immigration is restricted.

2. A small barbershop, operated by a single barber, has room for at most two customers. Potential customers arrive at a Poisson rate of three per hour. If they find that the shop is full, then they leave. The successive service times are independent exponential random variables with mean $1/4$ hour.
 - (a). What is the average number of customers in the shop?
 - (b). What is the proportion of potential customers that enter the shop?
 - (c). If the barber could work twice as fast, how much more business would she do?
3. If $\{X_t, t \geq 0\}$ and $\{Y_t, t \geq 0\}$ are independent time-reversible Markov chains, show that the process $\{(X_t, Y_t), t \geq 0\}$ is also.
4. Consider two M/M/1 queues with respective parameters λ_i, μ_i , where $\lambda_i < \mu_i, i = 1, 2$. Suppose that they both share the same waiting room, which has finite capacity N . Whenever this room is full and all potential arrivals to either queue are lost. Compute the limiting probability that there will be n people at the first queue (1 being served and $n - 1$ in the waiting room when $n > 0$) and m at the second.
(**Hint:** Use the result of Problem 3).