

Lecture 5: Renewal Theory I

0.1 Renewal Process

- A natural generalization of Poisson processes is to consider a counting process for which the interval times are independent and identically distributed. Such a counting process is called a *renewal process*. Formally, let $\{X_n, n = 1, 2, \dots\}$ be a sequence of nonnegative independent random variables with a common distribution F . To avoid trivialities suppose that $F(0) = P[X_n = 0] < 1$. X_n can be regarded as the interarrival time between the $(n-1)$ st and the n th event. Let

$$S_0 = 0, \quad S_n = \sum_{i=1}^n X_i, n \geq 1.$$

S_n is the time of the n th event. Denote $N(t)$ to be the number of events by time t , i.e.,

$$N(t) = \sup\{n : S_n \leq t\}.$$

$N(t)$ is called a renewal process. Assume that

$$\mu = E[X_n].$$

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$$N(t) \geq n \Leftrightarrow S_n \leq t.$$

From this, we obtain

$$P[N(t) = n] = P[N(t) \geq n] - P[N(t) \geq n+1] = P[S_n \leq t] - P[S_{n+1} \leq t]$$

Let $F_n(t)$ be the distribution function of S_n . Then, $P[N(t) = n] = F_n(t) - F_{n+1}(t)$.

Theorem 1 Let $m(t) = E[N(t)]$. We have

$$m(t) = \sum_{n=1}^{+\infty} F_n(t).$$

Theorem 2 $m(t) < +\infty$ for all $0 \leq t < +\infty$.

- **Example 1** A Poisson process is a renewal process.

Example 2 Let $Z_0, Z_1, \dots$ be a recurrent Markov chain. Suppose that $Z_0 = i$ and consider the times between successive visits to state i . Specifically, $X_1 = \min\{n > 0 : Z_n = i\}$ and $X_{k+1} = \min\{n > X_k : Z_n = i\} - X_k$. Then, $X_1, X_2, \dots$ are i.i.d. and thus we may define a renewal process based on them.

Example 3 In a single-server queue, there are imbedded many natural renewal processes. We cite one examples: if customer arrival times form a renewal process, then the times of the start of successive busy period generates a second renewal process.

0.2 Wald Equation

- Let $X_1, X_2, \dots$ denote a sequence of independent random variables. We define an integer-valued random variable N a *stopping time* for the sequence $X_1, X_2, \dots$ if the event $\{N(t) = n\}$ is independent of $X_{n+1}, X_{n+2}, \dots$ for all $n = 1, 2, \dots$. Intuitively, we observe the X_n 's in sequential order and N denotes the number observed before stopping. If $N = n$, then we have stopped after observing $X_1, \dots, X_n$ and before $X_{n+1}, X_{n+2}, \dots$.

Example 4 Let $X_n, n = 1, 2, \dots$ be independent and such that

$$P[X_n = 1] = P[X_n = -1] = 1/2.$$

Let $N = \min\{n : X_1 + \dots + X_n = 10\}$, then N is a stopping time.

- **Theorem 3** Let $X_1, X_2, \dots$ be i.i.d. random numbers with finite expectations. If N is a stopping time for $X_1, X_2, \dots$ such that $E[N] < +\infty$, then

$$E\left[\sum_{i=1}^N X_i\right] = E[N]E[X].$$

The condition $E[N] < +\infty$ is essential, as illustrated by the following example:

Example 5 Consider the setting of Example 4. $\sum_{i=1}^N X_i = 10$, but $E[X] = 0$. The Wald equation does not hold here. The reason is that $E[N] = +\infty$.

Example 6 Let $X_1, X_2, \dots$ be the interarrival times of a renewal process and let us stop at the first renewal after t — that is, at the $N(t) + 1$ renewal. It is easy to verify that $N(t) + 1$ is also a stopping time. Then,

$$E[S_{N(t)+1}] = E[X_1 + \dots + X_{N(t)+1}] = E[X] \cdot E[N(t) + 1] = \mu \cdot (m(t) + 1).$$

0.3 Some Limit Theorems

- $\lim_{t \rightarrow +\infty} N(t) = +\infty$ and

Theorem 4 (Law of Large Number) With probability 1,

$$\lim_{t \rightarrow +\infty} \frac{N(t)}{t} = \frac{1}{\mu}.$$

Example 7 Suppose that you have infinite supply of coins. Each coin has its own probability of landing heads, and these probabilities are the values of independent random variables that are uniformly distributed over $(0, 1)$. We flip coins sequentially, at any time either flipping a new coin or one that had previously been used. If the objective is to maximize the long-run proportion of flips that lands on heads, how should you do?

Theorem 5 (Central Limit Theorem) Let μ and σ^2 represent the mean and variance of an interarrival time. Then,

$$\lim_{t \rightarrow +\infty} P\left[\frac{N(t) - t/\mu}{\sigma\sqrt{t/\mu^3}} < y\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-x^2/2} dx.$$

• **Theorem 6 (Elementary Renewal Theorem)**

$$\lim_{t \rightarrow +\infty} \frac{m(t)}{t} = \frac{1}{\mu}.$$

0.4 Renewal Equation and Key Renewal Theorem

- Consider $m(t) = E[N(t)]$. Conditioning on the first arrival time of event, X_1 , we have

$$m(t) = E[E[N(t)|X_1]] = E[(1 + m(t - X_1)) \cdot \mathbf{1}_{\{X_1 \leq t\}}] = F(t) + \int_0^t m(t - y) dF(y).$$

An integral equation of the form

$$A(t) = a(t) + \int_0^t A(t - y) dF(y), \quad t \geq 0 \quad (1)$$

is called a *renewal equation*. Hence, $m(t)$ satisfies a renewal equation.

On the other hand, any renewal equation (1) can be solved in terms of $m(t)$. This result is summarized in the following theorem

Theorem 7 *If A satisfies the renewal equation (1), then A is given by*

$$A(t) = a(t) + \int_0^t a(t - y) dm(y).$$

Example 8 *Revisit Example 6. Let $A(t) = E[S_{N(t)+1}]$. Conditioning on $X_1 = x$,*

$$E[S_{N(t)+1}|X_1 = x] = \begin{cases} x, & \text{if } x > t; \\ x + A(t - x), & \text{if } x \leq t. \end{cases}$$

Then,

$$A(t) = \int_0^{+\infty} E[S_{N(t)+1}|X_1 = x] dF(x) = E[X_1] + \int_0^t A(t - x) dF(x).$$

By Theorem 7,

$$A(t) = E[X_1] \cdot [1 + m(t)].$$

- A nonnegative random variable X is said to be *lattice* or *arithmetic* if there exists $d \geq 0$ such that $\sum_{n=0}^{+\infty} P[X = nd] = 1$. We state the following theorem without proof.

Theorem 8 (Key Renewal Theorem) *Suppose that F is a distribution function of a positive random variable with mean μ and A is the solution of the following renewal equation*

$$A(t) = a(t) + \int_0^t A(t - y) dF(y).$$

(i). If F is not lattice, then

$$\lim_{t \rightarrow +\infty} A(t) = \begin{cases} \frac{1}{\mu} \int_0^{+\infty} a(x) dx, & \text{if } \mu < +\infty; \\ 0, & \text{if } \mu = +\infty. \end{cases}$$

(ii) If F is a lattice with span d , then

$$\lim_{n \rightarrow +\infty} A(c + nd) = \begin{cases} \frac{d}{\mu} \sum_0^{+\infty} a(c + nd), & \text{if } \mu < +\infty; \\ 0, & \text{if } \mu = +\infty. \end{cases}$$

Example 9 (Elementary renewal theorem revisited) *One direct application of Theorem 7 is to prove the elementary renewal theorem. In fact, let $h > 0$ and examine a special a such that*

$$a(t) = \begin{cases} 1, & \text{if } 0 \leq t < h; \\ 0, & \text{if } h \leq t. \end{cases}$$

Suppose that A satisfies the renewal equation

$$A(t) = a(t) + \int_0^t A(t-y)dF(y).$$

According to Theorem 8, if F is not lattice, $\lim_{t \rightarrow +\infty} A(t) = \mu^{-1} \int_0^{+\infty} a(x)dx = h/\mu$. On the other hand, by Theorem 7,

$$A(t) = a(t) + \int_0^t a(t-y)dm(y) = m(t) - m(t-h)$$

when $t > h$. Then we have

$$\lim_{t \rightarrow +\infty} [m(t) - m(t-h)] = h/\mu.$$

Furthermore, for a positive integer n ,

$$\frac{1}{n}m(n) = \frac{1}{n} \sum_{i=1}^n [m(i) - m(i-1)] \rightarrow \frac{h}{\mu}.$$