SEEM5020 2026 Spring: Homework 1

This assignment is due at 1:30 pm, January 16, 2026. Please bring your solution and submit it at the beginning of the next class. Total weight of Homework 1: 3%.

- **Q1. [10 Point] Coin Flipping in Rounds.** Assume that initially, we have n fair coins. We flip the coin in rounds. In the first round, we flip all of these n coins (once for each coin); if a coin shows the tail, it is removed. In the next round (Round 2), we have the coins that are left (which showed the head in the last round) and flip each of these coins. Again, if a coin shows the tail, it is removed. Then, we turn to Round 3, Round 4, etc. until there is no coin left. Answer the following questions.
  - I. [3 marks] What is the expected number of coins left at the end of round 3?
  - II. [3 marks] What is the expected number of coin tosses we have flipped in total (summing up tosses from all rounds)?
  - III. [4 marks] Prove that the expected number of rounds we have played is bounded by $\log_2 n+2$. You may use the tail sum formula for expectation: $\mathbb{E}[X] = \sum_{i=1}^{\infty} \mathbb{P}[X \geq i]$. Useful inequality: $(1+x)^r \geq 1 + x \cdot r$ for any $x \geq -1$ and $r > 1$.
- **Q2. [10 Point] Random Graphs.** Consider a random graph $G(n, p)$ with n vertices. For every pair of distinct vertices u, v, we add an undirected edge (u, v) independently with probability p.
  - I. [3 marks] What is the expected number of edges in G?
  - II. [3 marks] A triangle is a set of 3 distinct vertices $\{u, v, w\}$ such that edges (u, v), (v, w), and (u, w) all exist. What is the expected number of triangles in G?
  - III. [4 marks] An isolated vertex is a vertex with degree 0 (no edges connected to it). What is the expected number of isolated vertices in G?
- **Q3. [10 Point] Fixed Points in Permutation.** Let π be a permutation of $\{1, 2, \ldots, n\}$ chosen uniformly at random from all $n!$ possible permutations. A fixed point is an index i such that $\pi(i) = i$. Let X be the random variable representing the number of fixed points in π.
  - I. [5 marks] Calculate the expectation $\mathbb{E}[X]$.
  - II. [5 marks] Calculate the variance $\mathrm{Var}[X]$.