SEEM5020 2026 Spring: Homework 1

This assignment is due at 1:30 pm, January 16, 2026. Please bring your solution and submit it at the beginning of the next class. Total weight of Homework 1: 3%.

- **Q1. [10 Point] Coin Flipping in Rounds.** Assume that initially, we have n fair coins. We flip the coin in rounds. In the first round, we flip all of these n coins (once for each coin); if a coin shows the tail, it is removed. In the next round (Round 2), we have the coins that are left (which showed the head in the last round) and flip each of these coins. Again, if a coin shows the tail, it is removed. Then, we turn to Round 3, Round 4, etc. until there is no coin left. Answer the following questions.
  - I. [3 marks] What is the expected number of coins left at the end of round 3?
  - II. [3 marks] What is the expected number of coin tosses we have flipped in total (summing up tosses from all rounds)?
  - III. [4 marks] Prove that the expected number of rounds we have played is bounded by $\log_2 n + 2$. You may use the tail sum formula for expectation: $\mathbb{E}[X] = \sum_{i=1}^{\infty} \mathbb{P}[X \geq i]$. Useful inequality: $(1+x)^r \geq 1 + x \cdot r$ for any $x \geq -1$ and $r > 1$.
- **Q2. [10 Point] Random Graphs.** Consider a random graph $G(n, p)$ with n vertices. For every pair of distinct vertices u, v, we add an undirected edge (u, v) independently with probability p.
  - I. [3 marks] What is the expected number of edges in G?
  - II. [3 marks] A triangle is a set of 3 distinct vertices $\{u, v, w\}$ such that edges (u, v), (v, w), and (u, w) all exist. What is the expected number of triangles in G?
  - III. [4 marks] An isolated vertex is a vertex with degree 0 (no edges connected to it). What is the expected number of isolated vertices in G?
- **Q3. [10 Point] Fixed Points in Permutation.** Let π be a permutation of $\{1, 2, \ldots, n\}$ chosen uniformly at random from all $n!$ possible permutations. A fixed point is an index i such that $\pi(i) = i$. Let X be the random variable representing the number of fixed points in π.
  - I. [5 marks] Calculate the expectation $\mathbb{E}[X]$.
  - II. [5 marks] Calculate the variance $\mathrm{Var}[X]$.

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- **Q1**
  - **I.** Let X_i be an indicator variable that is 1 if the i-th coin remains after round 3, and 0 otherwise. A coin remains after round 3 if it shows Heads in Round 1, 2, and 3. Since the coin is fair and flips are independent:

$$\mathbb{P}(X_i = 1) = \left(\frac{1}{2}\right)^3 = \frac{1}{8}.$$

    Let X be the total number of coins left. By Linearity of Expectation:

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X_i] = n \cdot \frac{1}{8} = \frac{n}{8}.$$

  - **II.** Let T_i be the number of times the i-th coin is tossed. The coin is tossed in Round k if it survived Round $k-1$ (where "surviving Round 0" is true by default).

$$\mathbb{P}(\text{coin } i \text{ is tossed in Round } k) = \mathbb{P}(\text{coin } i \text{ survives } k-1 \text{ rounds}) = \left(\frac{1}{2}\right)^{k-1}.$$

    The expected number of tosses for coin i is:

$$\mathbb{E}[T_i] = \sum_{k=1}^{\infty} 1 \cdot \mathbb{P}(\text{tossed in Round } k) = \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{k-1} = 1 + \frac{1}{2} + \frac{1}{4} + \cdots = 2.$$

    Let T be the total tosses. By Linearity of Expectation:

$$\mathbb{E}[T] = \sum_{i=1}^{n} \mathbb{E}[T_i] = 2n.$$

  - **III.** Let Y be the number of rounds played. The game continues to round i if at least one coin survives round $i-1$. The probability that a specific coin survives round $i-1$ is $(1/2)^{i-1}$. The probability that a specific coin is removed by round $i-1$ is $1-(1/2)^{i-1}$. The probability that all coins are removed by round $i-1$ is $(1-(1/2)^{i-1})^n$. Thus, the probability that the game is still ongoing at round i (meaning $Y \geq i$) is:

$$\mathbb{P}(Y \geq i) = 1 - \left(1 - \frac{1}{2^{i-1}}\right)^n.$$

    Using the inequality $(1+x)^r \geq 1 + xr$ with $x = -1/2^{i-1}$ and $r = n$:

$$\left(1 - \frac{1}{2^{i-1}}\right)^n \geq 1 - \frac{n}{2^{i-1}}.$$

    Therefore, $\mathbb{P}(Y \geq i) \leq 1 - (1 - \frac{n}{2^{i-1}}) = \frac{n}{2^{i-1}}$. Also, trivially $\mathbb{P}(Y \geq i) \leq 1$.

Using the tail sum formula, we split the sum at $k = \lfloor \log_2 n \rfloor + 1$:

$$\begin{aligned}
\mathbb{E}[Y] &= \sum_{i=1}^{k} \mathbb{P}(Y \geq i) + \sum_{i=k+1}^{\infty} \mathbb{P}(Y \geq i) \\
&\leq \sum_{i=1}^{k} 1 + \sum_{i=k+1}^{\infty} \frac{n}{2^{i-1}} = k + n \sum_{j=k}^{\infty} \frac{1}{2^j} = k + n \left(\frac{1}{2^{k-1}} \right) \\
&= (\lfloor \log_2 n \rfloor + 1) + n \cdot \frac{1}{2^{\lfloor \log_2 n \rfloor}} \qquad (\text{Let } \log_2 n = \lfloor \log_2 n \rfloor + \{\log_2 n\}) \\
&= \log_2 n - \{\log n\} + 1 + n \cdot \frac{2^{\{\log_2 n\}}}{2^{\log_2 n}} \\
&= \log_2 n + 1 + 2^{\{\log_2 n\}} - \{\log_2 n\} \qquad (\text{Notice that } 2^x - x \leq 1 \text{ for } 0 \leq x < 1) \\
&\leq \log_2 n + 2.
\end{aligned}$$

- **Q2**
  - **I.** There are $\binom{n}{2}$ possible edges. Let X_{uv} be the indicator that edge (u, v) exists. $\mathbb{E}[X_{uv}] = p$.

$$\mathbb{E}[\#\text{Edges}] = \sum_{\{u,v\}} \mathbb{E}[X_{uv}] = \binom{n}{2} p = \frac{n(n-1)}{2} p.$$

  - **II.** There are $\binom{n}{3}$ possible sets of 3 vertices. For a set $\{u, v, w\}$ to form a triangle, edges $(u, v), (v, w), (u, w)$ must all exist. Since edges are independent:

$$\mathbb{P}(\{u, v, w\} \text{ form a triangle}) = p \cdot p \cdot p = p^3.$$

$$\mathbb{E}[\#\text{Triangles}] = \binom{n}{3} p^3 = \frac{n(n-1)(n-2)}{6} p^3.$$

  - **III.** A vertex v is isolated if it is not connected to any of the other $n - 1$ vertices.

$$\mathbb{P}(v \text{ is isolated}) = (1-p)^{n-1}.$$

$$\mathbb{E}[\#\text{Isolated Vertices}] = \sum_{v=1}^{n} \mathbb{P}(v \text{ is isolated}) = n(1-p)^{n-1}.$$

- **Q3**
  - **I.** Let X_i be the indicator variable that $X_i = 1$ if $\pi(i) = i$ and 0 otherwise. Since π is a random permutation, the probability that the i-th element is mapped to position i is $1/n$.

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X_i] = \sum_{i=1}^{n} \frac{1}{n} = n \cdot \frac{1}{n} = 1.$$

  - **II.** We use the formula $\text{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$. We know $\mathbb{E}[X] = 1$.

$$X^2 = (\sum_{i=1}^{n} X_i)^2 = \sum_{i=1}^{n} X_i^2 + \sum_{i \neq j} X_i X_j.$$

Since X_i is an indicator, $X_i^2 = X_i$, so $\sum \mathbb{E}[X_i^2] = \mathbb{E}[X] = 1$. For $i \neq j$, $X_i X_j = 1$ only if $\pi(i) = i$ AND $\pi(j) = j$.

$$\mathbb{E}[X_i X_j] = \mathbb{P}(\pi(i) = i \cap \pi(j) = j) = \frac{1}{n} \cdot \frac{1}{n-1} = \frac{1}{n(n-1)}.$$

There are $n(n-1)$ such pairs.

$$\mathbb{E}[\sum_{i \neq j} X_i X_j] = n(n-1) \cdot \frac{1}{n(n-1)} = 1.$$

Thus, $\mathbb{E}[X^2] = 1 + 1 = 2$.

$$\mathrm{Var}[X] = 2 - (1)^2 = 1.$$