SEEM5020 2026 Spring: Homework 6

This assignment is due at 1:30 pm, March 13, 2026. Please bring your solution and submit it at the beginning of the next class. Total weight of Homework 6: 3%.

- **Q1. [10 Point] Frobenius Norm and Singular Values**

  Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ be a matrix of rank r with singular value decomposition $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$, where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ are the singular values. Prove that:

  $$\|\mathbf{A}\|_F^2 = \sum_{i=1}^{r} \sigma_i^2.$$

  Hint: Recall that $\|\mathbf{A}\|_F^2 = \mathrm{Tr}(\mathbf{A}^T\mathbf{A})$ and use the cyclic property of the trace: $\mathrm{Tr}(\mathbf{XY}) = \mathrm{Tr}(\mathbf{YX})$.

- **Q2. [20 Point] Generalizing Truncated SVD for $k \leq r$**

  Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ have rank r. We want to find a rank-k matrix $\mathbf{B}$ ($k \leq r$) minimizing the reconstruction error $\|\mathbf{A} - \mathbf{B}\|_F^2$. Suppose $\mathbf{B}$ is written as

  $$\mathbf{B} = \sum_{j=1}^{k} \mathbf{x}_j \mathbf{y}_j^T,$$

  where $\mathbf{y}_1, \ldots, \mathbf{y}_k \in \mathbb{R}^n$ are mutually orthonormal unit vectors, and $\mathbf{x}_j \in \mathbb{R}^m$.

  (a) Prove that, after choosing the optimal vectors $\mathbf{x}_j$, the objective of minimizing $\|\mathbf{A} - \mathbf{B}\|_F^2$ reduces to

  $$\min_{\mathbf{y}_1,\ldots,\mathbf{y}_k} \left(\|\mathbf{A}\|_F^2 - \sum_{j=1}^{k} \|\mathbf{A}\mathbf{y}_j\|_2^2 \right).$$

  (b) Prove that the minimum reconstruction error is achieved by choosing $\mathbf{y}_1, \ldots, \mathbf{y}_k$ to be orthonormal eigenvectors of $\mathbf{A}^T\mathbf{A}$ corresponding to its k largest eigenvalues, and show that the minimum error is $\sum_{i=k+1}^{r} \sigma_i^2$, where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ are the singular values of $\mathbf{A}$.

  Hint for (a): Let $\mathbf{a}_i^T$ denote the i-th row of $\mathbf{A}$. For fixed $\mathbf{y}_j$'s, the optimal $\mathbf{x}_j$ is found by projecting each row onto the subspace spanned by $\mathbf{y}_1, \ldots, \mathbf{y}_k$.

  Hint for (b): Define $\mathbf{Y} = [\mathbf{y}_1, \ldots, \mathbf{y}_k]$ and consider the matrix $\mathbf{P} = \mathbf{V}^T\mathbf{Y}\mathbf{Y}^T\mathbf{V}$, where $\mathbf{A}^T\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^T$. What are the properties of $\mathbf{P}$?