SEEM5020 2026 Spring: Homework 6 Solution

- **Q1. Frobenius Norm and Singular Values**

  Solution: Since $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$, we have $\mathbf{A}^T\mathbf{A} = \mathbf{V}\mathbf{\Sigma}^T\mathbf{U}^T\mathbf{U}\mathbf{\Sigma}\mathbf{V}^T = \mathbf{V}\mathbf{\Sigma}^2\mathbf{V}^T$, where we used $\mathbf{U}^T\mathbf{U} = \mathbf{I}$.

  Then:

  $$\|\mathbf{A}\|_F^2 = \mathrm{Tr}(\mathbf{A}^T\mathbf{A}) = \mathrm{Tr}(\mathbf{V}\mathbf{\Sigma}^2\mathbf{V}^T) = \mathrm{Tr}(\mathbf{V}^T\mathbf{V}\mathbf{\Sigma}^2) = \mathrm{Tr}(\mathbf{\Sigma}^2) = \sum_{i=1}^{r}\sigma_i^2,$$

  where we used the cyclic property $\mathrm{Tr}(\mathbf{XYZ}) = \mathrm{Tr}(\mathbf{ZXY})$ and $\mathbf{V}^T\mathbf{V} = \mathbf{I}$.

- **Q2. Generalizing Truncated SVD for $k \le r$**

  Solution:

  (a) Let $\mathbf{a}_i^T$ denote the i-th row of $\mathbf{A}$, and let $x_{i,j}$ be the i-th entry of $\mathbf{x}_j$. The i-th row of $\mathbf{B}$ is $\sum_{j=1}^{k} x_{i,j}\mathbf{y}_j^T$. Therefore,

  $$\|\mathbf{A}-\mathbf{B}\|_F^2 = \sum_{i=1}^{m}\left\|\mathbf{a}_i - \sum_{j=1}^{k} x_{i,j}\mathbf{y}_j\right\|_2^2.$$

  For each i, since $\mathbf{y}_1,\ldots,\mathbf{y}_k$ are orthonormal, the minimum over $x_{i,1},\ldots,x_{i,k}$ is achieved when $\sum_j x_{i,j}\mathbf{y}_j$ is the orthogonal projection of $\mathbf{a}_i$ onto $\mathrm{span}\{\mathbf{y}_1,\ldots,\mathbf{y}_k\}$. Hence $x_{i,j} = \mathbf{a}_i^T\mathbf{y}_j$, or equivalently $\mathbf{x}_j = \mathbf{A}\mathbf{y}_j$.

  By the Pythagorean theorem and orthonormality of $\mathbf{y}_j$'s:

  $$\left\|\mathbf{a}_i - \sum_{j=1}^{k}(\mathbf{a}_i^T\mathbf{y}_j)\mathbf{y}_j\right\|_2^2 = \|\mathbf{a}_i\|_2^2 - \sum_{j=1}^{k}(\mathbf{a}_i^T\mathbf{y}_j)^2.$$

  Summing over all rows and noting that $\sum_{i=1}^{m}\|\mathbf{a}_i\|_2^2 = \|\mathbf{A}\|_F^2$ and $\sum_{i=1}^{m}(\mathbf{a}_i^T\mathbf{y}_j)^2 = \|\mathbf{A}\mathbf{y}_j\|_2^2$:

  $$\|\mathbf{A}-\mathbf{B}\|_F^2 = \|\mathbf{A}\|_F^2 - \sum_{j=1}^{k}\|\mathbf{A}\mathbf{y}_j\|_2^2.$$

  (b) From part (a), minimizing $\|\mathbf{A}-\mathbf{B}\|_F^2$ is equivalent to maximizing $\sum_{j=1}^{k}\|\mathbf{A}\mathbf{y}_j\|_2^2$. Let $\mathbf{Y} = [\mathbf{y}_1,\ldots,\mathbf{y}_k] \in \mathbb{R}^{n\times k}$ with $\mathbf{Y}^T\mathbf{Y} = \mathbf{I}_k$. Then:

  $$\sum_{j=1}^{k}\|\mathbf{A}\mathbf{y}_j\|_2^2 = \sum_{j=1}^{k}\mathbf{y}_j^T\mathbf{A}^T\mathbf{A}\mathbf{y}_j = \mathrm{Tr}(\mathbf{Y}^T\mathbf{A}^T\mathbf{A}\mathbf{Y}).$$

  Let $\mathbf{A}^T\mathbf{A} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^T$ be the eigendecomposition with $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0$. By the cyclic property of trace:

  $$\mathrm{Tr}(\mathbf{Y}^T\mathbf{A}^T\mathbf{A}\mathbf{Y}) = \mathrm{Tr}(\mathbf{\Lambda}\cdot\mathbf{V}^T\mathbf{Y}\mathbf{Y}^T\mathbf{V}).$$

  Define $\mathbf{P} = \mathbf{V}^T\mathbf{Y}\mathbf{Y}^T\mathbf{V}$. Since $\mathbf{Y}\mathbf{Y}^T$ is an orthogonal projection matrix and $\mathbf{V}$ is orthogonal, $\mathbf{P}$ is also an orthogonal projection matrix: $\mathbf{P}^2 = \mathbf{P}$, $\mathbf{P}^T = \mathbf{P}$, $\mathrm{Tr}(\mathbf{P}) = k$, and $0 \le P_{ii} \le 1$ for all i.

Therefore:

$$\mathrm{Tr}(\mathbf{\Lambda P}) = \sum_{i=1}^{n} \lambda_i P_{ii}, \quad \text{where } 0 \leq P_{ii} \leq 1 \text{ and } \sum_{i=1}^{n} P_{ii} = k.$$

Since $\lambda_1 \geq \cdots \geq \lambda_n$, this is maximized by setting $P_{ii} = 1$ for $i = 1, \ldots, k$ and $P_{ii} = 0$ otherwise, giving max $= \sum_{i=1}^{k} \lambda_i$. This is attained by choosing $\mathbf{y}_j = \mathbf{v}_j$ (the j-th eigenvector of $\mathbf{A}^T\mathbf{A}$). Since $\lambda_i = \sigma_i^2$:

$$\min \|\mathbf{A} - \mathbf{B}\|_F^2 = \|\mathbf{A}\|_F^2 - \sum_{i=1}^{k} \sigma_i^2 = \sum_{i=1}^{r} \sigma_i^2 - \sum_{i=1}^{k} \sigma_i^2 = \sum_{i=k+1}^{r} \sigma_i^2.$$