SEEM5020 2026 Spring: Homework 8 Solution

- **Q1. Existence of an η-Net on the Unit Sphere.**

  Solution:

  (a) Existence. We construct $\mathcal{N}$ via a greedy algorithm. Pick any $\mathbf{v}_1 \in \mathcal{S}$ and set $\mathcal{N} = \{\mathbf{v}_1\}$. Repeat: if there exists $\mathbf{u} \in \mathcal{S}$ such that $\|\mathbf{u} - \mathbf{v}\|_2 > \eta$ for all $\mathbf{v} \in \mathcal{N}$, add $\mathbf{u}$ to $\mathcal{N}$. Since $\mathcal{S}$ is compact (closed and bounded in $\mathbb{R}^r$), this process terminates in finitely many steps.

  By construction, when the process terminates, for every $\mathbf{u} \in \mathcal{S}$ there exists $\mathbf{v} \in \mathcal{N}$ with $\|\mathbf{u} - \mathbf{v}\|_2 \leq \eta$. Hence $\mathcal{N}$ is an internal η-net of $\mathcal{S}$.

  (b) Size bound. By the greedy construction, the points in $\mathcal{N}$ are pairwise at distance greater than η. Consider open balls $B(\mathbf{v}, \eta/2)$ of radius $\eta/2$ centered at each $\mathbf{v} \in \mathcal{N}$, all within the r-dimensional subspace $\mathcal{U}$.

  Disjointness: If $\mathbf{v}_i, \mathbf{v}_j \in \mathcal{N}$ with $i \neq j$, then $\|\mathbf{v}_i - \mathbf{v}_j\|_2 > \eta$, so $B(\mathbf{v}_i, \eta/2) \cap B(\mathbf{v}_j, \eta/2) = \emptyset$.

  Containment: For any $\mathbf{v} \in \mathcal{N}$ and $\mathbf{x} \in B(\mathbf{v}, \eta/2)$, by the triangle inequality:

  $$\|\mathbf{x}\|_2 \leq \|\mathbf{v}\|_2 + \|\mathbf{x} - \mathbf{v}\|_2 < 1 + \eta/2.$$

  Thus $B(\mathbf{v}, \eta/2) \subseteq B(\mathbf{0}, 1 + \eta/2)$ for all $\mathbf{v} \in \mathcal{N}$.

  Since the disjoint balls are all contained in $B(\mathbf{0}, 1+\eta/2)$, comparing r-dimensional volumes:

  $$|\mathcal{N}| \cdot \mathrm{Vol}_r(\eta/2) \leq \mathrm{Vol}_r(1 + \eta/2).$$

  Using $\mathrm{Vol}_r(R) = C_r \cdot R^r$:

  $$|\mathcal{N}| \leq \frac{C_r(1 + \eta/2)^r}{C_r(\eta/2)^r} = \left(\frac{1 + \eta/2}{\eta/2}\right)^r = \left(1 + \frac{2}{\eta}\right)^r.$$

- **Q2. Low-Rank Approximation via JL Transformation.**

  Solution:

  Trace decomposition. We first show $\|\mathbf{A} - \hat{\mathbf{A}}\|_F^2 = \|\mathbf{A}\|_F^2 - \|\mathbf{A}\mathbf{V}_k\|_F^2$. Since $\mathbf{V}_k$ has orthonormal columns ($\mathbf{V}_k^T\mathbf{V}_k = \mathbf{I}_k$):

  $$\begin{aligned}
  \|\mathbf{A} - \hat{\mathbf{A}}\|_F^2 = \|\mathbf{A} - \mathbf{A}\mathbf{V}_k\mathbf{V}_k^T\|_F^2 &= \mathrm{Tr}\left((\mathbf{A} - \mathbf{A}\mathbf{V}_k\mathbf{V}_k^T)(\mathbf{A} - \mathbf{A}\mathbf{V}_k\mathbf{V}_k^T)^T\right) \\
  &= \mathrm{Tr}\left(\mathbf{A}\mathbf{A}^T - 2\mathbf{A}\mathbf{V}_k\mathbf{V}_k^T\mathbf{A}^T + \mathbf{A}\mathbf{V}_k\underbrace{\mathbf{V}_k^T\mathbf{V}_k}_{\mathbf{I}_k}\mathbf{V}_k^T\mathbf{A}^T\right) \\
  &= \mathrm{Tr}(\mathbf{A}\mathbf{A}^T) - \mathrm{Tr}(\mathbf{A}\mathbf{V}_k\mathbf{V}_k^T\mathbf{A}^T) = \|\mathbf{A}\|_F^2 - \|\mathbf{A}\mathbf{V}_k\|_F^2.
  \end{aligned}$$

  Connecting $\|\mathbf{A}\mathbf{V}_k\|_F^2$ to $\sum_{i=1}^k \sigma_i^2(\mathbf{B})$. We show $\left|\|\mathbf{A}\mathbf{V}_k\|_F^2 - \sum_{i=1}^k \sigma_i^2(\mathbf{B})\right| \leq \sqrt{k}\|\mathbf{A}^T\mathbf{A} -$

$\mathbf{B}^T\mathbf{B}\|_F$. Note that $\sigma_i^2(\mathbf{B}) = \|\mathbf{B}\mathbf{b}_i\|_2^2$ (since $\mathbf{b}_i$ is the i-th right singular vector of $\mathbf{B}$).

$$\begin{aligned}\left|\|\mathbf{A}\mathbf{V}_k\|_F^2 - \sum_{i=1}^k \sigma_i^2(\mathbf{B})\right| &= \left|\sum_{i=1}^k \left(\|\mathbf{A}\mathbf{b}_i\|_2^2 - \|\mathbf{B}\mathbf{b}_i\|_2^2\right)\right| \\ &\le \sum_{i=1}^k \left|\|\mathbf{A}\mathbf{b}_i\|_2^2 - \|\mathbf{B}\mathbf{b}_i\|_2^2\right| \\ &\le \sqrt{k}\sqrt{\sum_{i=1}^k \left(\|\mathbf{A}\mathbf{b}_i\|_2^2 - \|\mathbf{B}\mathbf{b}_i\|_2^2\right)^2} \quad \text{(Cauchy–Schwarz)}\end{aligned}$$

Since $\|\mathbf{A}\mathbf{b}_i\|_2^2 - \|\mathbf{B}\mathbf{b}_i\|_2^2 = \mathbf{b}_i^T(\mathbf{A}^T\mathbf{A} - \mathbf{B}^T\mathbf{B})\mathbf{b}_i$:

$$\begin{aligned}&= \sqrt{k}\sqrt{\sum_{i=1}^k \left(\mathbf{b}_i^T(\mathbf{A}^T\mathbf{A} - \mathbf{B}^T\mathbf{B})\mathbf{b}_i\right)^2} \\ &\le \sqrt{k}\sqrt{\sum_{i=1}^k \|\mathbf{b}_i\|_2^2\|(\mathbf{A}^T\mathbf{A} - \mathbf{B}^T\mathbf{B})\mathbf{b}_i\|_2^2} \quad \text{(Cauchy–Schwarz)} \\ &= \sqrt{k}\sqrt{\sum_{i=1}^k \mathbf{b}_i^T(\mathbf{A}^T\mathbf{A} - \mathbf{B}^T\mathbf{B})^T(\mathbf{A}^T\mathbf{A} - \mathbf{B}^T\mathbf{B})\mathbf{b}_i} \\ &\le \sqrt{k}\|\mathbf{A}^T\mathbf{A} - \mathbf{B}^T\mathbf{B}\|_F.\end{aligned}$$

The last step uses the fact that $\mathbf{b}_1, \ldots, \mathbf{b}_k$ are orthonormal: denoting $\mathbf{M} = (\mathbf{A}^T\mathbf{A} - \mathbf{B}^T\mathbf{B})$,

$$\sum_{i=1}^k \mathbf{b}_i^T\mathbf{M}^T\mathbf{M}\mathbf{b}_i \le \sum_{i=1}^n \mathbf{b}_i^T\mathbf{M}^T\mathbf{M}\mathbf{b}_i = \mathrm{Tr}(\mathbf{V}^T\mathbf{M}^T\mathbf{M}\mathbf{V}) = \mathrm{Tr}(\mathbf{M}^T\mathbf{M}) = \|\mathbf{M}\|_F^2.$$

Connecting $\sum_{i=1}^k \sigma_i^2(\mathbf{B})$ to $\sum_{i=1}^k \sigma_i^2(\mathbf{A})$. We show $\left|\sum_{i=1}^k \sigma_i^2(\mathbf{B}) - \sum_{i=1}^k \sigma_i^2(\mathbf{A})\right| \le \sqrt{k}\|\mathbf{A}^T\mathbf{A} - \mathbf{B}^T\mathbf{B}\|_F$. By Cauchy–Schwarz:

$$\left|\sum_{i=1}^k \sigma_i^2(\mathbf{B}) - \sum_{i=1}^k \sigma_i^2(\mathbf{A})\right| \le \sqrt{k}\sqrt{\sum_{i=1}^k \left(\sigma_i^2(\mathbf{B}) - \sigma_i^2(\mathbf{A})\right)^2}.$$

Since $\sigma_i^2(\mathbf{B}) = \sigma_i(\mathbf{B}^T\mathbf{B})$ and $\sigma_i^2(\mathbf{A}) = \sigma_i(\mathbf{A}^T\mathbf{A})$:

$$\sqrt{\sum_{i=1}^k \left(\sigma_i(\mathbf{B}^T\mathbf{B}) - \sigma_i(\mathbf{A}^T\mathbf{A})\right)^2} \le \sqrt{\sum_{i=1}^n \left(\sigma_i(\mathbf{B}^T\mathbf{B}) - \sigma_i(\mathbf{A}^T\mathbf{A})\right)^2} \le \|\mathbf{B}^T\mathbf{B} - \mathbf{A}^T\mathbf{A}\|_F,$$

where the last inequality is the Hoffman–Wielandt inequality.

Combining the bounds. Using $\|\mathbf{A} - \mathbf{A}_k\|_F^2 = \|\mathbf{A}\|_F^2 - \sum_{i=1}^k \sigma_i^2(\mathbf{A})$:

$$\begin{aligned}\|\mathbf{A} - \hat{\mathbf{A}}\|_F^2 &= \|\mathbf{A}\|_F^2 - \|\mathbf{A}\mathbf{V}_k\|_F^2 \\ &\le \|\mathbf{A}\|_F^2 + \sqrt{k}\|\mathbf{A}^T\mathbf{A} - \mathbf{B}^T\mathbf{B}\|_F - \sum_{i=1}^k \sigma_i^2(\mathbf{B}) \\ &\le \|\mathbf{A}\|_F^2 + 2\sqrt{k}\|\mathbf{A}^T\mathbf{A} - \mathbf{B}^T\mathbf{B}\|_F - \sum_{i=1}^k \sigma_i^2(\mathbf{A}) \\ &= \|\mathbf{A} - \mathbf{A}_k\|_F^2 + 2\sqrt{k}\|\mathbf{A}^T\mathbf{A} - \mathbf{B}^T\mathbf{B}\|_F.\end{aligned}$$

- **Q3. JLT Preserves All Directions in a Subspace.**

  Solution:

  (a)(i) Each coordinate j_i ranges over $\{0, 1, \ldots, 2d/\varepsilon\}$, which contains $2d/\varepsilon + 1$ values. Since $1/\varepsilon$ is an integer, $2d/\varepsilon$ is a positive integer. With d independent coordinates, the total number of grid points is $\left(\frac{2d}{\varepsilon} + 1\right)^d$.

  (a)(ii) For any $\mathbf{z} \in \mathcal{S}$, since $\|\mathbf{z}\|_2 = 1$, each coordinate satisfies $|z_i| \leq 1$, so $\mathbf{z} \in [-1, 1]^d$. The grid partitions $[-1, 1]^d$ into cells of side length ε/d, so $\mathbf{z}$ lies inside some grid cell. Let $\mathbf{g}$ be the nearest grid vertex. In each coordinate, $|z_i - g_i| \leq \varepsilon/(2d)$, so:

$$\|\mathbf{z} - \mathbf{g}\|_2 = \sqrt{\sum_{i=1}^{d} (z_i - g_i)^2} \leq \sqrt{d \cdot \frac{\varepsilon^2}{4d^2}} = \frac{\varepsilon}{2\sqrt{d}}.$$

  (b)(i) For each coordinate i: since 0 is a grid point and the grid has spacing ε/d, the grid points in dimension i include $0, \pm\varepsilon/d, \pm 2\varepsilon/d, \ldots$ Rounding z_i toward zero gives the grid value g_i^- satisfying $|g_i^-| \leq |z_i|$ (we move from z_i toward 0, hitting a grid point no further from 0 than z_i). Therefore $\|\mathbf{g}^-\|_2^2 = \sum_i (g_i^-)^2 \leq \sum_i z_i^2 = 1$.

  (b)(ii) Rounding z_i away from zero gives g_i^+ with $|g_i^+| \geq |z_i|$. This is valid since $|z_i| \leq 1$ and ± 1 are grid points, so there is always a grid point at least as far from zero. Therefore $\|\mathbf{g}^+\|_2^2 = \sum_i (g_i^+)^2 \geq \sum_i z_i^2 = 1$.

  (b)(iii) For either choice $\mathbf{g} \in \{\mathbf{g}^-, \mathbf{g}^+\}$, $\mathbf{g}$ is a vertex of the cell containing $\mathbf{z}$, so $|\Delta_i| = |z_i - g_i| \leq \varepsilon/d$. Write $\Delta_i = \alpha_i \cdot (\varepsilon/d) \cdot s_i$ where $\alpha_i = |z_i - g_i|/(\varepsilon/d) \in [0, 1]$ and $s_i = \text{sign}(z_i - g_i) \in \{-1, +1\}$. Then:

$$\Delta = \sum_{i=1}^{d} \alpha_i \cdot \frac{\varepsilon}{d} s_i \, \mathbf{e}_i.$$

  Each vector $\frac{\varepsilon}{d} s_i \, \mathbf{e}_i$ equals $(\mathbf{g} + \frac{\varepsilon}{d} s_i \, \mathbf{e}_i) - \mathbf{g}$, the difference between two adjacent grid points.

  (c) Since $\mathbf{\Pi}$ satisfies JLT$(\varepsilon, \delta, |\mathcal{G}|)$, the JLT property holds for all pairwise differences of grid points. In particular, for any grid point $\mathbf{v}$ (the origin is a grid point, so the difference $\mathbf{v} - \mathbf{0} = \mathbf{v}$ is preserved):

$$(1 - \varepsilon)\|\mathbf{v}\|_2^2 \leq \|\mathbf{\Pi}\mathbf{v}\|_2^2 \leq (1 + \varepsilon)\|\mathbf{v}\|_2^2.$$

  Taking square roots and using $\sqrt{1 + \varepsilon} \leq 1 + \varepsilon$ and $\sqrt{1 - \varepsilon} \geq 1 - \varepsilon$:

$$(1 - \varepsilon)\|\mathbf{v}\|_2 \leq \|\mathbf{\Pi}\mathbf{v}\|_2 \leq (1 + \varepsilon)\|\mathbf{v}\|_2.$$

  Similarly, each grid step $\frac{\varepsilon}{d} s_i \mathbf{e}_i$ is a difference of adjacent grid points, so:

$$\left\| \mathbf{\Pi} \left(\frac{\varepsilon}{d} s_i \, \mathbf{e}_i \right) \right\|_2 \leq (1 + \varepsilon) \frac{\varepsilon}{d}.$$

  Bounding the shift. By part (b)(iii) and the triangle inequality:

$$\|\mathbf{\Pi}\Delta\|_2 \leq \sum_{i=1}^{d} \alpha_i \left\| \mathbf{\Pi} \left(\frac{\varepsilon}{d} s_i \, \mathbf{e}_i \right) \right\|_2 \leq d \cdot (1 + \varepsilon) \frac{\varepsilon}{d} = \varepsilon(1 + \varepsilon) = \varepsilon + \varepsilon^2.$$

  Upper bound. Write $\mathbf{z} = \mathbf{g}^- + \Delta$. By the triangle inequality and part (b)(i):

$$\|\mathbf{\Pi}\mathbf{z}\|_2 \leq \|\mathbf{\Pi}\mathbf{g}^-\|_2 + \|\mathbf{\Pi}\Delta\|_2 \leq (1 + \varepsilon) \underbrace{\|\mathbf{g}^-\|_2}_{\leq 1} + (\varepsilon + \varepsilon^2) \leq 1 + 2\varepsilon + \varepsilon^2.$$

Since $\varepsilon < 1/3$, we have $\varepsilon^2 < \varepsilon$, so $\|\mathbf{\Pi z}\|_2 < 1 + 3\varepsilon = (1+3\varepsilon)\|\mathbf{z}\|_2$.

Lower bound. Write $\mathbf{z} = \mathbf{g}^+ - \Delta'$. By the reverse triangle inequality and part (b)(ii):

$$\|\mathbf{\Pi z}\|_2 \geq \|\mathbf{\Pi g}^+\|_2 - \|\mathbf{\Pi}\Delta'\|_2 \geq (1-\varepsilon)\underbrace{\|\mathbf{g}^+\|_2}_{\geq 1} - (\varepsilon + \varepsilon^2) \geq 1 - 2\varepsilon - \varepsilon^2.$$

Since $\varepsilon^2 < \varepsilon$, we have $\|\mathbf{\Pi z}\|_2 > 1 - 3\varepsilon = (1-3\varepsilon)\|\mathbf{z}\|_2$.