SEEM5020 2026 Spring: Homework 9

This assignment is due at 1:30 pm, April 10, 2026. Please bring your solution and submit it at the beginning of the next class. Total weight of Homework 9: 2%.

- **Q1. [10 Points] Covariance Error Bound for Frequent Directions.**

  In the lecture (Theorem 3), we showed that the sketch $\mathbf{B}$ returned by Frequent Directions with sketch size ℓ satisfies, for every $\mathbf{x} \in \mathbb{R}^d$,

  $$0 \leq \|\mathbf{A}\mathbf{x}\|_2^2 - \|\mathbf{B}\mathbf{x}\|_2^2 \leq \frac{\|\mathbf{A}\|_F^2}{\ell}\|\mathbf{x}\|_2^2.$$

  Prove the equivalent matrix form:

  (a) **[5 Points]** $\mathbf{A}^T\mathbf{A} - \mathbf{B}^T\mathbf{B} \succeq \mathbf{0}$ (i.e., $\mathbf{A}^T\mathbf{A} - \mathbf{B}^T\mathbf{B}$ is positive semidefinite).

  (b) **[5 Points]** $\|\mathbf{A}^T\mathbf{A} - \mathbf{B}^T\mathbf{B}\|_2 \leq \frac{\|\mathbf{A}\|_F^2}{\ell}$, where $\|\cdot\|_2$ denotes the spectral norm (largest singular value).

  Hint: For (a), recall that a symmetric matrix $\mathbf{M}$ is PSD if and only if $\mathbf{x}^T\mathbf{M}\mathbf{x} \geq 0$ for all $\mathbf{x}$. For (b), recall that for a symmetric PSD matrix $\mathbf{M}$, $\|\mathbf{M}\|_2 = \lambda_{\max}(\mathbf{M}) = \max_{\mathbf{x}\neq\mathbf{0}} \frac{\mathbf{x}^T\mathbf{M}\mathbf{x}}{\|\mathbf{x}\|_2^2}$.

- **Q2. [10 Points] Low-Rank Approximation via Frequent Directions.**

  Let $\mathbf{B}$ be the sketch returned by Frequent Directions with sketch size ℓ. Let $\mathbf{V}_k$ be the matrix whose columns are the top-k right singular vectors of $\mathbf{B}$, and define the projection $\mathbf{P}_{\mathbf{B}_k} = \mathbf{V}_k\mathbf{V}_k^T$.

  Using Theorem 4 (which states that for any $k < \ell$, $0 \leq \|\mathbf{A}\mathbf{x}\|_2^2 - \|\mathbf{B}\mathbf{x}\|_2^2 \leq \frac{\|\mathbf{A}-\mathbf{A}_k\|_F^2}{\ell-k}\|\mathbf{x}\|_2^2$), prove that:

  $$\|\mathbf{A} - \mathbf{A}\mathbf{P}_{\mathbf{B}_k}\|_F^2 \leq \left(1 + \frac{k}{\ell - k}\right)\|\mathbf{A} - \mathbf{A}_k\|_F^2,$$

  where $\mathbf{A}_k$ is the best rank-k approximation of $\mathbf{A}$.

  Hint: Write $\|\mathbf{A} - \mathbf{A}\mathbf{P}_{\mathbf{B}_k}\|_F^2 = \|\mathbf{A}\|_F^2 - \sum_{j=1}^k \|\mathbf{A}\mathbf{v}_j\|_2^2$, where $\mathbf{v}_j$ are the top-k right singular vectors of $\mathbf{B}$. Use Theorem 4 to relate $\|\mathbf{A}\mathbf{v}_j\|_2^2$ to $\sigma_j^2(\mathbf{B})$, then use the variational characterization of singular values to relate $\sum_{j=1}^k \sigma_j^2(\mathbf{B})$ to $\sum_{j=1}^k \sigma_j^2(\mathbf{A})$.