SEEM5020 2026 Spring: Homework 9 Solution

- **Q1. Covariance Error Bound for Frequent Directions.**

  Solution:

  (a) Let $\mathbf{M} = \mathbf{A}^T\mathbf{A} - \mathbf{B}^T\mathbf{B}$. Note that $\mathbf{M}$ is symmetric. For any $\mathbf{x} \in \mathbb{R}^d$:

  $$\mathbf{x}^T\mathbf{M}\mathbf{x} = \mathbf{x}^T(\mathbf{A}^T\mathbf{A} - \mathbf{B}^T\mathbf{B})\mathbf{x} = \|\mathbf{A}\mathbf{x}\|_2^2 - \|\mathbf{B}\mathbf{x}\|_2^2 \geq 0,$$

  where the inequality follows from the left-hand side of the given bound. Since $\mathbf{x}^T\mathbf{M}\mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^d$, $\mathbf{M}$ is positive semidefinite.

  (b) Since $\mathbf{M} = \mathbf{A}^T\mathbf{A} - \mathbf{B}^T\mathbf{B}$ is symmetric PSD (from part (a)), its spectral norm equals its largest eigenvalue:

  $$\|\mathbf{M}\|_2 = \lambda_{\max}(\mathbf{M}) = \max_{\mathbf{x}\neq\mathbf{0}} \frac{\mathbf{x}^T\mathbf{M}\mathbf{x}}{\|\mathbf{x}\|_2^2}.$$

  From the right-hand side of the given bound:

  $$\mathbf{x}^T\mathbf{M}\mathbf{x} = \|\mathbf{A}\mathbf{x}\|_2^2 - \|\mathbf{B}\mathbf{x}\|_2^2 \leq \frac{\|\mathbf{A}\|_F^2}{\ell}\|\mathbf{x}\|_2^2.$$

  Dividing both sides by $\|\mathbf{x}\|_2^2$ and taking the maximum over all $\mathbf{x} \neq \mathbf{0}$:

  $$\|\mathbf{M}\|_2 = \max_{\mathbf{x}\neq\mathbf{0}} \frac{\mathbf{x}^T\mathbf{M}\mathbf{x}}{\|\mathbf{x}\|_2^2} \leq \frac{\|\mathbf{A}\|_F^2}{\ell}.$$

- **Q2. Low-Rank Approximation via Frequent Directions.**

  Solution:

  Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be the top-k right singular vectors of $\mathbf{B}$, and let $\mathbf{u}_1, \ldots, \mathbf{u}_k$ be the top-k right singular vectors of $\mathbf{A}$.

  Step 1: Express the approximation error. Since $\mathbf{V}_k$ has orthonormal columns:

  $$\|\mathbf{A} - \mathbf{A}\mathbf{P}_{\mathbf{B}_k}\|_F^2 = \|\mathbf{A} - \mathbf{A}\mathbf{V}_k\mathbf{V}_k^T\|_F^2 = \|\mathbf{A}\|_F^2 - \|\mathbf{A}\mathbf{V}_k\|_F^2 = \|\mathbf{A}\|_F^2 - \sum_{j=1}^{k}\|\mathbf{A}\mathbf{v}_j\|_2^2.$$

  Step 2: Relate $\|\mathbf{A}\mathbf{v}_j\|_2^2$ to $\sigma_j^2(\mathbf{B})$. By Theorem 4, for each unit vector $\mathbf{v}_j$:

  $$\|\mathbf{A}\mathbf{v}_j\|_2^2 \geq \|\mathbf{B}\mathbf{v}_j\|_2^2 = \sigma_j^2(\mathbf{B}).$$

  Summing over $j = 1, \ldots, k$:

  $$\sum_{j=1}^{k}\|\mathbf{A}\mathbf{v}_j\|_2^2 \geq \sum_{j=1}^{k}\sigma_j^2(\mathbf{B}).$$

  Step 3: Relate $\sum_{j=1}^{k}\sigma_j^2(\mathbf{B})$ to $\sum_{j=1}^{k}\sigma_j^2(\mathbf{A})$. By the variational characterization of singular values:

  $$\sum_{j=1}^{k}\sigma_j^2(\mathbf{B}) = \max_{\mathbf{W}\in\mathbb{R}^{d\times k},\, \mathbf{W}^T\mathbf{W}=\mathbf{I}_k} \|\mathbf{B}\mathbf{W}\|_F^2 \geq \sum_{j=1}^{k}\|\mathbf{B}\mathbf{u}_j\|_2^2.$$

  Denote $\Delta = \frac{\|\mathbf{A}-\mathbf{A}_k\|_F^2}{\ell-k}$. By Theorem 4, for each unit vector $\mathbf{u}_j$:

  $$\|\mathbf{B}\mathbf{u}_j\|_2^2 \geq \|\mathbf{A}\mathbf{u}_j\|_2^2 - \Delta = \sigma_j^2(\mathbf{A}) - \Delta.$$

Therefore:

$$\sum_{j=1}^{k} \sigma_j^2(\mathbf{B}) \geq \sum_{j=1}^{k} \left(\sigma_j^2(\mathbf{A}) - \Delta\right) = \sum_{j=1}^{k} \sigma_j^2(\mathbf{A}) - k\Delta.$$

Step 4: Combine. Putting Steps 1–3 together and using $\|\mathbf{A} - \mathbf{A}_k\|_F^2 = \|\mathbf{A}\|_F^2 - \sum_{j=1}^{k} \sigma_j^2(\mathbf{A})$:

$$\begin{aligned}
\|\mathbf{A} - \mathbf{A}\mathbf{P}_{\mathbf{B}_k}\|_F^2 &= \|\mathbf{A}\|_F^2 - \sum_{j=1}^{k} \|\mathbf{A}\mathbf{v}_j\|_2^2 \\
&\leq \|\mathbf{A}\|_F^2 - \sum_{j=1}^{k} \sigma_j^2(\mathbf{A}) + k\Delta \\
&= \|\mathbf{A} - \mathbf{A}_k\|_F^2 + k \cdot \frac{\|\mathbf{A} - \mathbf{A}_k\|_F^2}{\ell - k} \\
&= \left(1 + \frac{k}{\ell - k}\right) \|\mathbf{A} - \mathbf{A}_k\|_F^2.
\end{aligned}$$