Q1. [5 Point] Let $X$ be a non-negative integer-valued random variable with $E[X] < \infty$. Prove that:

$$E[X] = \sum_{i=1}^{\infty} P[X \geq i].$$

Q2. [5 Point] Consider again the Quiz question (Page 18, Lec 1). Use the conclusion of Q1 in this assignment and prove that the expected number of rounds we have played is bounded by $\log n + 2$. Useful inequality: $(1 + x)^r \geq 1 + x \cdot r$ for any $x \geq -1$ and $r > 1$.

Q3. [5 Point] Given $n$ balls and $n$ bins, assume that we independently throw these $n$ balls and that each ball has the same chance of being thrown into these $n$ bins. What is the expected number of bins that include exactly 1 ball? What is the expected number of bins that include exactly $i$ balls ($i \leq n$)? Hint: you might need to use $\binom{n}{k}$.

Q4. [5 Point] Given a binary vector of size $n$ (initially all zero), we mark the binary vector in $m$ rounds. For each round, we randomly select $k$ positions in the array and mark it as 1 (Notice that we might mark the same position as 1). What is the expected number of zero entries?

Example: Given a binary vector of size 8. If we run in $m = 2$ rounds and $k$ is 2. Then, in the first iteration, we randomly choose two positions. Assume that it is position 2 and position 2. Then the binary vector becomes $[0, 1, 0, 0, 0, 0, 0, 0]$. In the second round, we randomly select two positions and assume that they are 3 and 5. Then, the array becomes $[0, 1, 1, 0, 1, 0, 0, 0]$. The number of zero entries is 5.