Electronic Market Making and Latency

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Abstract

This paper studies the profitability of market making strategies and the impact of latency on electronic market makers' profits for large–tick assets. By analyzing the optimal market making problem using Markov Decision Processes, we provide simple conditions to determine when a market maker earns zero or positive profits and discuss economic implications. We also prove that higher latency leads to reduced profits for market makers, and conduct numerical experiments to illustrate the effect of latency and relative latency on the market maker's expected profit. Finally, our work highlights the importance of value of orders in optimal market making.

1 Introduction

A market maker in a security market provides liquidity to other investors by quoting bids and offers, hoping to make a profit from the bid-ask spread while avoiding accumulating a large net position in the assets traded. Market makers play a crucial role in financial markets, as the liquidity they offer allows investors to obtain immediate executions of their orders, and this flexibility facilitates market efficiency and functioning. Traditionally in equity markets, there are "official" or designated market makers who have entered into contractual agreements with exchanges and they are under certain affirmative obligations to stand ready to supply liquidity. In recent decades, major financial markets have became electronic, and a modern exchange is usually operated as an electronic limit order book system where all the outstanding limit orders are aggregated for market participants to view [23]. As a result, any professional trader can adopt market making as a trading strategy, often through computer-based electronic trading decisions and automated computer-based trade executions. Such traders are called electronic market makers as in [26]. They are not obliged and can enter and exit market at will. This work focuses on such "unofficial" electronic market makers.

This paper studies optimal market making in a limit order book and try to answer two questions for large-tick assets: (1) when market making strategies are profitable? (2) how does the latency affects the profitability of market making strategies? Large-tick assets are those assets with large relative tick size (the dollar tick size, e.g. one cent, normalized by the price of the asset) and their bid-ask spreads rarely exceed one tick [8, 9]. Empirically,

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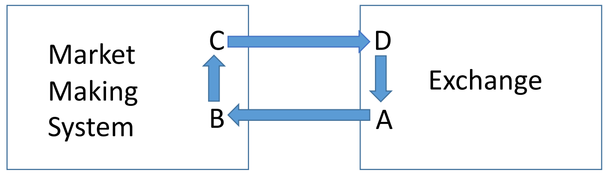

Figure 1: Latency: the entire cycle $A \to B \to C \to D \to A$

it has been found that electronic (high-frequency) market makers take on a prominent role in liquidity provision for large-tick stocks [24]. This is because the revenue margins for liquidity provision are higher for those low-price stocks with larger relative tick sizes [29].

The two research questions we focus on are clearly important for trading firms who perform market making strategies. For example, the solutions to these questions may serve as guidelines for high-level decisions such as market entrance as a liquidity provider and investment in IT infrastructure to reduce latency. They are also potentially relevant for regulators and policy makers to understand when these electronic market makers may withdraw from liquidity provision, assuming negative profits cause these firms to exit.

There are many different definitions for the term "latency" in the literature. We follow [15] to define the latency a market maker experiences and use Figure 1 to illustrate. The total time delay for the cycle $A \to B \to C \to D \to A$ is defined as the latency in our study. Here, from point A to B, updates of market information including asset prices and order status are sent from the exchange to the market maker. From B to C, market analysis is then performed by the trading algorithms, and decisions such as canceling old quotes and sending new quotes are made. From Point C to D, the market maker's actions are sent to the exchange, and finally from D to A, the matching engine in the exchange accepts the orders, processes them, confirms them and possibly executes them. This entire cycle can be as low as several milliseconds for some high-frequency market makers. In general, the latency an electronic market maker experiences depends on many factors including the physical distance between the server running the market making algorithm and the matching engine of the exchange, the mean of access by the trader to the exchange (e.g. direct market access or via a retail platform), the complexity of the market making algorithm, and the trading system used by each exchange. In this paper, as in [21, 28], we assume a constant latency $\Delta\tau \geq 0$ for tractability purposes, though in practice the latency is random [18].

Low latency is important for market makers. It enables market makers to respond rapidly to newly available price information and changing market conditions by placing or canceling orders [21]. In fact, some high-frequency market makers spend millions each year to place servers on which their trading algorithms run as close as possible to the exchange's matching engine. This co-location allows data to travel at a minimal distance (e.g., $A \to B$ in Figure 1) and allows market makers to get ultra low latency access to an exchange's trading and information systems [16].

1.1 Contributions

In this paper, we study the profitability of electronic market making strategies for large-tick assets in the presence of latency. We consider an "unofficial" market maker who can quote a bid order for one unit and an ask order for one unit at any discrete prices periodically with a deterministic period length $\Delta t > 0$, with the goal of maximizing his expected profit within a finite horizon. The market maker determines the optimal buy and sell prices at each period, which is formalized as a finite-horizon Markov Decision Process (MDP) [4, 25]. By a delicate analysis of the MDP, we make the following contributions.

First, we provide explicit conditions to determine when a market maker earns zero or positive expected profits[3] (Theorem 3). Under our model, the main criteria is simply to compare two rates: the rate of change in the asset price and the rate of "uninformed" market order flows that hit the market maker's limit orders at best quotes. The latter depends on the relative latency of the market maker, i.e., how fast the market maker is relative to others in the competition for front queue positions to obtain execution priority at desired price points. Our result suggests that in certain scenarios such as the market is volatile or the market maker is not fast enough in gaining good queue positions compared with others, then electronic market making on the single asset is not profitable, regardless of how low the absolute latency $\Delta\tau$ the market maker experiences. This illustrates the importance of relative latency for market makers and it implies the potential low-latency arms race among market makers. It also implies that these unofficial market makers may withdraw from liquidity provision during volatile and stressed market times.

Second, our study sheds light on how latency affects the profits of market makers. We prove that, holding other parameters fixed, the expected profit of the market maker decreases as the absolute latency $\Delta\tau$ increases (Proposition 4). Latency is an additional source of risk for market makers due to the possible price motion in the latency time window. High latency increases the chances that the prices of market maker's quotes are crossed[4] by the mid-price. It also increases the chances of one-sided[5] fills of the market maker's bid-ask pair of orders, hence the inventory risk the market maker bears may increase. We also conduct numerical experiments based on real data to show the significance of low latency (both in absolute and relative terms) and its impact on the optimal quoting strategy of a market maker.

[3] Fixed overhead costs related to market data feed, trading infrastructures etc, are not considered in the profit calculation here.

[4] For example, the market maker observes the market best bid is currently at \$10.00 and he sends a buy limit order at \$10.00. By the time this order reaches the exchange, the best offer (bid) has already become \$10.00 (\$9.99), and this order is filled.

[5] For example, if the market price jumps up during the latency period, then the fill probability of the market maker's ask order increases and that of the bid order decreases.

Third, our work highlights the importance of value of an order in optimal market making. The value of an order measures the difference between the execution price of the order and the 'fundamental value' of the asset. This quantity, often referred to as the expected profit of an order, has been widely used in equilibrium models in the finance and economics literature. For example, an endogenous bid-ask spread of an asset can be produced by assuming perfectly competitive markets makers earning zero profits in equilibrium, see e.g. [17, 27] and the references therein. However, the order value has not been adequately explored in the literature on optimal market making. Our work bridges this gap by showing that the value of bid and ask orders sent by the market maker in each period essentially plays the role of one-period reward in the MDP model (Theorem 2). This provides further insights into the optimal market making problem. It also facilitates our analysis as whether market making is profitable hinges on whether the value of the orders sent by the market maker is positive.

1.2 Literature review

In this section, we explain the differences between our work and the existing studies.

There has been a number of studies on optimal market making in the quantitative finance literature, see, e.g., [1, 2, 5, 6, 7, 10, 12, 13] and the references therein. These papers mainly use continuous-time stochastic control approaches to determine optimal (continuous) quoting strategies for market makers in an expected utility framework. Our work complements these studies but differs from them in two main aspects: (1) we explicitly take latency into account in optimal market making; and (2) our main focus is to understand when market making leads to positive profits and the effect of latency on the market maker's profit.

The second line of literature that is related to our work is the study of latency, or broadly speed, in algorithm and high frequency trading. In [21], the authors propose a model to quantify the cost of latency on optimal trade execution of one share, but not on market making. See also [28] for a similar related work. In addition, the work [20] studies the latency inside the exchange, and they found speeding up the exchange does not necessarily improve liquidity. Furthermore, the empirical study [3] finds that differences in relative latency account for large differences in performances of high frequency trading firms (not necessarily electronic market makers). Finally, a number of related studies have investigated the relation between speed, trading and market quality. For a comprehensive review of this literature, we refer the readers to [19]. Our work differs from these studies in that we focus on electronic market making and the effect of latency on optimal market making strategies.

The rest of the paper is organized as follows. In Section 2 we describe the model and formulate the market maker's optimization problem using Markov Decision Processes. In Section 3, we present the main theoretical results on the profitability of market making strategies and the effect of latency on the profit. In Section 4, we present numerical results based on parameters estimated from order book data for a representative stock. Finally, Section 5 concludes. Some technical details of the model and all the proofs of the theoretical results are given in the Appendix.

2 The MDP model for market making

In the section we formulate the market making problem as a finite-horizon and discrete-time Markov Decision Process (MDP). We consider an electronic market maker maximizing his expected terminal wealth (equivalently, net profit) within all admissible policies. All stochastic processes and random variables are defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

2.1 Market making process

We now discuss the process of market making in the presence of latency.

We first describe the dynamics of asset price and order flows. We assume that the bid-ask spread of the asset is exogenously given at constant one tick, which is typical for large-tick liquid stocks [8]. We also assume that the best bid price $\{p(t) : t \geq 0\}$ is a compound Poisson process which jumps one tick (cent) at exponential times:

$$p(t) = p(0) + \sum_{i=1}^{\mathcal{N}(t)} X_i, \tag{2.1}$$

where $\{\mathcal{N}(t) : t \geq 0\}$ is a Poisson process with a rate λ and $(X_i)_{i=1,2,\dots}$ are independent and identically distributed random variables taking values $+1$ and -1 both with probability 0.5. See, e.g., [1, 14] for similar models. We use symmetric jump size X_i as market makers typically have no directional opinion on the assets they trade.

Next, we describe the market maker's periodic quoting process in the presence of latency. Figure 1 gives a graphical illustration. During a finite horizon $[0, T]$, the market maker takes actions $N+1$ times by canceling old/outstanding quotes and sending new bids and offers at *discrete* prices every Δt time units. The market maker's orders experience a constant absolute latency $\Delta\tau \geq 0$, and for the tractability of the model, we assume $\Delta\tau < \Delta t$. Starting from time zero, the exchange sends messages continuously to the market maker. The messages contain the asset price, information of the trader's wealth/cash, the inventory in the asset and the trader's outstanding orders (if any). Outstanding orders refer to limit orders (sent by the trader) waiting to be filled in the limit order book. The maker sends orders $N+1$ times in equal time intervals. In each period, except for the last one, the market maker sends a ask-bid quote pair and a cancellation instruction. The size of the quote is fixed at one unit (e.g., 100 shares) and the trading amount is called unit dollars. The cancellation instruction will automatically cancel the maker's any outstanding orders (if any). In our model, the cancellation instruction is required in each period

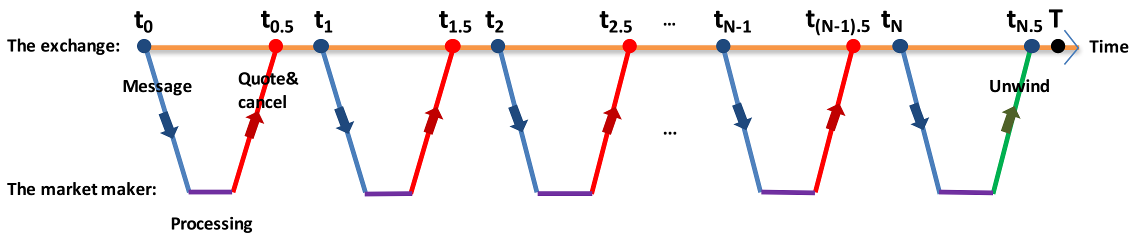

Figure 2: An illustration for the trader's market making process

to ensure the trader has at most one outstanding ask/bid order in the order book at any time. For the last period, the maker unwinds all his inventory using a market order.

We define the $N+1$ action times as the time when the message is sent by the exchange on which the corresponding actions are made, i.e.,

$$t_i := i \cdot \Delta t, \quad \text{for} \quad i = 0, 1, 2, ..., N. \tag{2.2}$$

So the quoting duration Δt also represents the life-time of the market maker's limit order if it is not executed. Due to latency, the time when the orders of i-th action enter into the limit order book is $t_{i.5} := t_i + \Delta\tau$. In particular, the time when the unwinding market order arrives at the exchange is

$$t_{N.5} := N \cdot \Delta t + \Delta\tau. \tag{2.3}$$

In the round-trip latency $\Delta\tau$, the proportions of there three parts (message to the maker, data processing, new quotes to exchange) do not matter in our model. Thus, for simplicity, throughout paper we say an order is sent at time t to mean that the order is based on the information at time t. We assume that the maker quotes as many times as possible. As $t_{N.5} \leq T$, we have $N = \lfloor \frac{T-\Delta\tau}{\Delta t} \rfloor$, where $\lfloor x \rfloor$ is the greatest integer that is smaller or equal to $x \in \mathbb{R}$.

As standard in the literature (see e.g. [12, 6]), to control the inventory risk, we assume the market maker's inventory (number of units of an asset held) is constrained by a lower bound $\underline{q}$ and a upper bound $\overline{q}$. Here $\underline{q}$ and $\overline{q}$ are two fixed integers with $\underline{q} < -1$ and $\overline{q} > 1$.

2.2 Order executions

We now describe the executions of the market maker's (limit) orders in our model. For illustrations, we consider an ask order. Considering a bid order is similar. In our setting, when the market maker's ask order arrives at the exchange, it can execute in one of the following ways:

1. If the limit price of an ask order is smaller or equal to the market best bid price, the order will get executed immediately.

2. Otherwise, the ask order will enter into the order book and it will get executed at its limit price if

   (a) either when the order sits at the best ask price and the best bid price jumps up by one tick, crossing its limit price;

   (b) or the total time the order spends at the best ask price exceeds an exponential random variable with rate λ^a.

Case (1) can occur, for example, when the maker sends a limit ask order and the mid-price of the asset moves up during the latency period. The execution price of the ask order is the market best bid price at the moment the ask order reaches the exchange.

In Case 2(a), the ask order is filled when the market trades through its limit price, and so the mid-price moves up one tick. This can happen if there is a large flow (e.g. a surge of buy orders into best ask) against the ask order. In Case 2(b), the mid-price does not move when the ask order is filled. We interpret λ^a as the rate of "uninformed" buy market order flow that matches with this market maker's ask order at the best ask, though we do not explicitly model information asymmetry among market participants. We remark that λ^a depends on both the rate of total "uninformed" buy market order arrivals and the relative latency (i.e., speed advantage compared with other traders) of the market maker, where the latter determines the queue position and execution priority of the order sent by the market maker. Similarly, we use λ^b for the rate of "uninformed" sell market order flow that matches with this market maker's buy order at the best bid price. These "uninformed" market order arrivals are assumed to be independent of the price process.

Mathematically, for period $i = 0, 1, ..., N$, we use two indicator functions $\mathbb{1}_{ask_i}$ and $\mathbb{1}_{bid_i}$ to specify whether the outstanding ask and bid orders (if any) at time t_i- are filled (1 if filled; 0 if not) before time $t_{i.5}$. Note these outstanding orders will be canceled at time $t_{i.5}$ by the market maker if they are not filled. Similarly, for $i = 0, 1, 2, ...N-1$, we use two indicator functions $\mathbb{1}_{ask_{i.5}}$ and $\mathbb{1}_{bid_{i.5}}$ to specify whether the (new) ask and bid orders sent by the market maker at time t_i- are filled before t_{i+1}. See Appendix A.4 for the formulas of these indicator functions for order executions.

2.3 State space and admissible action space

We now describe the system state and the admissible action space for the MDP model.

We write the state of available information at t as $s(t)$, and the set of extended integers as $\overline{\mathbb{Z}} := \mathbb{Z} \cup \{\pm\infty\}$. At any time $t \in [0, T]$, the exchange sends a message to the market maker containing information $s(t) := (w(t), p(t), q(t), a(t), b(t)) \in \mathbb{Z}^3 \times \overline{\mathbb{Z}}^2$, where $w(t)$ is the market maker's wealth, $p(t)$ is the best bid price of the asset, $q(t)$ is the maker's inventory. In addition, $(a(t), b(t))$ represents the outstanding ask-bid quote pair. More precisely, $p(t) + a(t)$ is the price of the marker's outstanding ask order at time t with $a(t) = \infty$ meaning there is no outstanding ask order at time t. Similarly, $p(t) + b(t)$ is the price of the marker's outstanding bid order at time t where $b(t) = -\infty$ means there is no bid order at time t. From the Poisson assumptions, it is easy to see that the sample paths of $s(t)$ are right-continuous with left limits. For $i = 0, 1, 2, ..., N$, denote

$$s_i = (w_i, p_i, q_i, a_i, b_i) := s(t_i-), \tag{2.4}$$

and

$$s_{i.5} = (w_{i.5}, p_{i.5}, q_{i.5}, a_{i.5}, b_{i.5}) := s(t_{i.5}-). \tag{2.5}$$

The $N + 1$ states $s_i, i = 0, 1, ..., N$ are corresponding to the maker's $N + 1$ actions (quote, cancellation or unwinding) and $s_{N.5}$ is the final state that is just before the time to exit market making. These $N+2$ states $s_0, s_1, ..., s_N, s_{N.5}$ are the system states for the discrete-time MDP and $s_{i.5}, i = 1, 2, ..., N - 1$, are intermediates to compute dynamics of these system states[6], see Sections 2.4. The $N + 1$ times t_i, $i = 0, 1, .., N$, can be called the decision epochs in the discrete-time MDP (though at t_N, there are no real decisions).

We can now write down the state space. Clearly we have $a(t) \geq 1$ and $b(t) \leq 0$. Note that when the maker's inventory reaches the lower/upper bound, he should not have any outstanding ask/bid orders at each decision epoch, since otherwise the inventory may exceed the two bounds due to possible execution of those outstanding orders. Hence, the state space S is given as follows.

$$\begin{aligned} S := \{(w, p, q, a, b) :&(w, p, q) \in \mathbb{Z}^3, (a, b) \in \overline{\mathbb{Z}} \times \overline{\mathbb{Z}}, \\ &\underline{q} \leq q \leq \overline{q}, \quad a \geq 1, \quad b \leq 0, \\ &\text{if } q = \underline{q}, \quad \text{then } a = \infty, \\ &\text{if } q = \overline{q}, \quad \text{then } b = -\infty\}. \end{aligned} \tag{2.6}$$

Next, we describe the admissible action space. When the market maker receives the system state $s = (w, p, q, a, b)$, he quotes an ask-bid pair at price $(p + \delta^a, p + \delta^b)$ together with an instruction to cancel his previous outstanding orders. The maker is allowed to send market orders, limit orders or not send any orders. Specifically, since the price is discrete, we have $(\delta^a, \delta^b) \in \overline{\mathbb{Z}} \times \overline{\mathbb{Z}}$, where $\delta^a = +\infty$ means no sell order is sent and $\delta^a = -\infty$ means a sell market order is sent. Similarly, $\delta^b = -\infty$ means no buy order is sent and $\delta^b = +\infty$ means a buy market order. We use A_s to denote the set of admissible actions (δ^a, δ^b) such that the inventory of the market maker always stays in the interval $[\underline{q}, \overline{q}]$ at any time. For the detailed mathematical expression of A_s, see Appendix A.2.

To set up the MDP, we still need to describe the dynamics for the system state. We refer the readers to Appendix A.3 for details.

[6]We use the left limits of the underlying continuous-time state process for the discrete-time state, which is a convention in the continuous-time stochastic control literature.

2.4 Optimization problem for the market maker

In this section, we formulate the optimization problem for the market maker. The maker quotes bid and ask orders at each period, and aims to maximize his expected terminal wealth after he unwinds the position at the end of the trading horizon. Costs of trading such as IT and compliance are not considered.

We first give the expression for the market maker's terminal wealth, denoted as TW. Suppose just before $t_{N.5}$, the state $s_{N.5} = s(t_{N.5}-)$ is $(w_{N.5}, p_{N.5}, q_{N.5}, a_{N.5}, b_{N.5})$. Then it is easy to see that

$$TW := w_{N.5} + p_{N.5} \cdot q_{N.5} + q_{N.5} \cdot \mathbb{1}_{q_{N.5}<0} = w_{N.5} + (p_{N.5} + 0.5)q_{N.5} - 0.5\,|q_{N.5}|\,. \tag{2.7}$$

That is, if the market maker as positive inventory $q_{N.5} > 0$, then the maker unwinds the position by sending a market sell order and the execution price is the best bid price $p_{N.5}$. Similarly, if $q_{N.5} < 0$, then the market maker sends a market buy order which will be filled at the best ask price $p_{N.5} + 1$. We do not consider the price impact of such a clean-up trade. This is reasonable as long as the market maker's inventory bounds do not exceed the market depth of best quotes in the order book. For large-tick assets, it is typical to find that there are large volumes of limit orders sitting at best quotes.

Now we can formulate the optimization problem of the market maker as follows:

$$v_0(s) = v_0(w,p,q,a,b) := \sup_{\pi} E^{\pi}[TW \mid s_0 = (w,p,q,a,b)], \tag{2.8}$$

where the supremum is taken over all Markovian admissible policies. Specifically, we have each Markov policy $\pi = (f_0, f_1, ..., f_N)$, where $f_i(\cdot)$ is a mapping from S to $\overline{\mathbb{Z}} \times \overline{\mathbb{Z}}$, such that for all $s \in S$, $f_i(s) \in A_s$, the admissible action space. This function v_0 is called the value function starting from the 0-th period. We can also define the value function starting from i-th period, $i = 1, 2, ..., N, N.5$, as follows:

$$\begin{aligned} &v_i(w,p,q,a,b) := \sup_{\pi\in\Pi} E^{\pi}[TW \mid s_i = (w,p,q,a,b)],\ i = 1,2,...,N, \\ &v_{N.5}(w,p,q,a,b) := w + pq + q\mathbb{1}_{q<0} = w + (p+0.5)q - 0.5\,|q|\,. \end{aligned} \tag{2.9}$$

Mathematically, in Equations (2.8) and (2.9), the existence of expectations is not trivial since the TW is not bounded. To address this issue, there is a standard method using an integrable bounding function to bound value functions. For our MDP, one can use the bounding function $C(|w| + |p| + 1)$ which can be verified to be integrable, where C is a constant that is independent of the state s. For simplicity, we omit the proof and refer readers to [4] for this method. We also remark that we do not include rebate or fee for orders in our model. However, our model can be readily generalized to include constant rebate and fee structure (for providing and taking liquidity respectively), and we can obtain similar results.

2.5 The Bellman equation

As we have formulated the market making problem as MDP, standard arguments show the following Bellman equation for the value functions:

$$v_i(s) = \begin{cases} w + (p+0.5)q - 0.5\,|q|\,, & i = N.5, \\ E[v_{N.5}(s_{N.5}) \mid s_N = s], & i = N, \\ \sup\limits_{(\delta^a,\delta^b)\in A_s} E^{(\delta^a,\delta^b)}[v_{i+1}(s_{N+1}) \mid s_i = s], & i = 0, 1, ...N-1, \end{cases} \tag{2.10}$$

where the superscript (δ^a, δ^b) for v_i means that the i-th action is (δ^a, δ^b). One can readily prove using the theory of upper semi-continuous MDP that the supremum operators in Equation (2.10) can be attained, which generates an optimal policy for the MDP. As the argument is standard (see, e.g., [4]), we omit the proof.

3 Market maker profitability and effects of latency

In this section, we present the main theoretical results on determining when the market making strategy is profitable and how latency affects the market maker's performance.

To be specific, the performance we consider is the net profit NP of the market maker, defined as

$$NP := v_0(w, p, 0, \infty, -\infty) - w. \tag{3.1}$$

That is, NP is the expected net wealth change over the horizon $[0, T]$, where the market maker starts with cash w, zero inventory ($q = 0$) and no outstanding orders ($a = \infty, b = -\infty$) in the limit order book at time zero. Our goal is to understand when NP is positive and how latency affects NP.

To this end, we first define the value of an order in Section 3.1. This quantity plays a critical role in understanding the structure of value functions discussed in Section 3.2, the profitability of market making strategies in Section 3.3 and the effect of latency in Section 3.4.

3.1 Value of an order

The value (or the expected profit) of an order essentially measures the difference of its execution price with the 'fundamental value' of the asset. For example, if there is no latency and one uses the asset mid-price at the time of order execution as the fundamental value, then the value of a market order is -0.5 ticks. That is, a trader pays half of the bid-ask spread using a market order.

To define the value of a general limit order when there is latency, we note that the market price might have moved between the moment an order sent by the market maker and the confirmed placement of an order. With this observation, we now first define the value of an ask order. Suppose at time zero, the ask order quoted at the price $p(0)+x$ with relative price $x \in \overline{\mathbb{Z}}$ is sent to the exchange. This order experiences a time delay $t'_1 \geq 0$ before its placement is confirmed by the exchange. We then compare the execution price of this order with the mid-price at time $t'_1 + t'_2$ which is regarded as the fundamental value of the asset. If the order is not executed, then the value of this order is zero.

Mathematically, for any $t'_1, t'_2 \geq 0$, we can define the value of an ask order with relative price x as follows:

$$H^{ask}(t'_1, t'_2, x) := E[(\max\{x - \Delta p[0, t'_1], 0\} - 0.5 - \Delta p[t'_1, t'_1 + t'_2]) \cdot \mathbb{1}_{ask_{t'_1, t'_2, x}}]. \tag{3.2}$$

Here, $\Delta p[t', t''] := p(t'') - p(t')$ indicates the change of the best bid price over the time interval $[t', t'']$. The execution price of this ask order is its limit price $p(0) + x$, or the market best bid price at the time t'_1, depending on whether the market best bid becomes higher than the limit price of the ask order when the ask order reaches the exchange. In addition, the mid-price at time $t'_1 + t'_2$ is given by $p(0) + \Delta p[0, t'_1] + \Delta p[t'_1, t'_1 + t'_2] + 0.5$ as the mid-price is half tick higher than the market best bid price. Finally, the indicator function $\mathbb{1}_{ask_{t'_1, t'_2, x}}$ specifies whether the ask order (which enters into the order book or executed at time t'_1) is filled before time $t'_1 + t'_2$. See Appendix A.4 for its mathematical expression.

The value of a buy order can be defined similarly. Suppose the buy limit order is quoted with price $p(0) + y$. Then for any $t'_1, t'_2 \geq 0$ and $y \in \overline{\mathbb{Z}}$, define the value of such a bid order sent at time 0 with relative price y with delay t'_1 and comparison time $t'_1 + t'_2$ as:

$$H^{bid}(t'_1, t'_2, y) := E[(0.5 + \Delta p[t'_1, t'_1 + t'_2] - \min\{y - \Delta p[0, t'_1], 1\}) \cdot \mathbb{1}_{bid_{t'_1, t'_2, y}}], \tag{3.3}$$

where $\mathbb{1}_{bid_{t'_1, t'_2, y}}$ indicates whether the bid order is filled before time $t'_1 + t'_2$.

Finally, if a market maker sends a pair of one bid and one ask orders, with relative prices $(x, y) \in \overline{\mathbb{Z}}^2$, then the value of this pair of quotes can be defined as follows: for any $t'_1, t'_2 \geq 0$,

$$H(t'_1, t'_2, x, y) := H^{ask}(t'_1, t'_2, x) + H^{bid}(t'_1, t'_2, y). \tag{3.4}$$

These functions will be used in understanding the MDP and the value functions for market making. Before we proceed, we first present a result to better understand these values of orders in our problem.

Proposition 1. *For any $\Delta\tau \geq 0$, $\Delta t > 0$ and $(\delta^a, \delta^b) \in \overline{\mathbb{Z}}^2$, we have*

(a)

$$H^{ask}(0, \Delta t, \delta^a) = \begin{cases} (\dfrac{\lambda^a}{\lambda^a + \lambda/2} - 0.5) E[\mathbb{1}_{ask_{0,\Delta t,\delta^a}}], & \delta^a \geq 1, \\ -0.5, & \delta^a \leq 0, \end{cases} \tag{3.5}$$

and

$$H^{bid}(0, \Delta t, \delta^b) = \begin{cases} (\dfrac{\lambda^b}{\lambda^b + \lambda/2} - 0.5) E[\mathbb{1}_{bid_{0,\Delta t,\delta^b}}], & \delta^b \leq 0, \\ -0.5, & \delta^b \geq 1. \end{cases} \tag{3.6}$$

(b)

$$H^{ask}(\Delta\tau, \Delta t, \delta^a) = E[H^{ask}(0, \Delta t, \delta^a - \Delta p[0, \Delta\tau])],$$

and

$$H^{bid}(\Delta\tau, \Delta t, \delta^b) = E[H^{bid}(0, \Delta t, \delta^b - \Delta p[0, \Delta\tau])].$$

We first discuss the economic interpretations of Part (a) of this result. For illustrations, we take $H^{ask}(0, \Delta t, \delta^a)$ as an example. For $\delta^a \le 0$, the ask order is effectively a market order, and will be filled instantly at the current best bid price as there is no latency. So its execution price is 0.5 tick less than the mid price at the time of execution. As the best bid price and the mid-price is a martingale with independent increments, it follows that the expected profit or the value of such an order on $[0, \Delta t]$ is -0.5 ticks. On the other hand, for $\delta^a \ge 1$, the limit sell order enters into the order book at time zero, and by (3.5) its value equals to a constant $\frac{\lambda^a}{\lambda^a+\lambda/2} - 0.5$ multiplied by the fill probability of the ask order $E[\mathbb{1}_{ask_{0,\Delta t,\delta^a}}]$ on $[0, \Delta t]$. This constant $\frac{\lambda^a}{\lambda^a+\lambda/2} - 0.5$ represents the conditional expected profit of the ask order given that the order is filled. To see this, we note that when such an ask order is filled, there are two scenarios: first, the ask order sits at the best ask price, and eventually transacts with an "uninformed" buy order, gaining 0.5 tick as the mid-price does not move at the time of execution ('spread capture'); second, the best bid price of the asset jumps up and crosses the quoted price of the ask order, in which case, the ask order loses 0.5 tick as the mid price immediately moves up one tick at the time of execution of the order ('adverse selection'). The rate of the first scenario occurs is λ^a, while the rate of the second scenario occurs is $\lambda/2$. Hence, the conditional expected profit the limit sell order is $\frac{\lambda^a}{\lambda^a+\lambda/2} \cdot 0.5 + \frac{\lambda}{\lambda^a+\lambda/2} \cdot (-0.5) = \frac{\lambda^a}{\lambda^a+\lambda/2} - 0.5$. While our model does not feature information asymmetry, the result here is generally consistent with the study [22] where they also find that one can informally interpret the order value as follows:

$$\text{Value of an order} = \text{fill probability} \times (\text{spread capture} - \text{adverse selection cost}).$$

We next discuss Part (b). It suggests that the value of orders with latency $\Delta\tau$ is the expected value of orders with zero latency where the quotes (δ^a, δ^b) are perturbed by random fluctuations of the market price during the latency window.

3.2 Structure of value functions

With the definitions of the value of orders H in (3.4), we now present the result on the structure of value functions. Recall that $\mathbb{1}_{ask_0}$, $\mathbb{1}_{bid_0}$ indicate whether the outstanding ask and bid orders are filled during $[0, \Delta\tau]$.

Theorem 2. *For any $s = (w, p, q, a, b) \in S$, we have $v_{N.5}(s) = w + (p + 0.5)q - 0.5|q|$ and*

$$v_i(s) = w + (p + 0.5)q + H(0, \Delta\tau, a, b) + g_i(q, a, b), \quad i = 0, 1, 2, ..., N, \tag{3.7}$$

where

$$g_i(q,a,b) := \begin{cases} -0.5E[\,|q - \mathbb{1}_{ask_0} + \mathbb{1}_{bid_0}|\,\big|(a_0,b_0) = (a,b)], & i = N, \\ \max\limits_{(\delta^a,\delta^b)\in A_s} G_i(q,a,b,\delta^a,\delta^b), & i = 0,1,...,N-1, \end{cases} \tag{3.8}$$

and

$$\begin{aligned} &G_i(q,a,b,\delta^a,\delta^b) := H(\Delta\tau,\Delta t,\delta^a,\delta^b) + E[g_{i+1}(q_1,a_1,b_1) \\ &\mid (q_0,a_0,b_0) = (q,a,b), (\delta^a_0,\delta^b_0) = (\delta^a,\delta^b)], \quad i = 0,1,...,N-1. \end{aligned} \tag{3.9}$$

Theorem 2 reduces the computation of value functions from five state-variables to three state-variables (q,a,b) in the backward recursion (3.8). As suggested by (3.7), the value function $v_i(w,p,q,a,b)$ can be decomposed into four parts: (1) w represents the market maker's current wealth or cash; (2) $(p+0.5)q$ is the value of the inventory marked to the market at the mid-price; (3) $H(0,\Delta\tau,a,b)$ is the value of the outstanding ask and bid orders as they will be canceled after $\Delta\tau$ units of time; and (4) $g_i(q,a,b)$ represents the extra value from following the optimal market making strategy. The backward recursion and maximization problem in (3.8) specifies the trade off between the value of the current actions/quotes (δ^a,δ^b) which depends on latency $\Delta\tau$, and the expected extra value g_{i+1} at the next period.

We also explain the similarities and differences between the structure of the value functions here and that in the existing literature on optimal market making with zero latency (under different models). When there is zero latency, i.e. $\Delta\tau = 0$, one can readily show that the value function does not depend on the prices of outstanding orders (a,b). This is because when $\Delta\tau = 0$, the outstanding orders at time t_i- will be canceled at time t_i instantly and hence they are not filled. In this case, the decomposition structure of value functions in (3.7) is similar as in [5]. Due to the presence of latency in our model, we can observe two main differences between our result and those in the literature (see e.g., [5]). First, our value functions include the value of the outstanding orders $H(0,\Delta\tau,a,b)$, where these outstanding orders will be canceled $\Delta\tau$ time units after the market maker receives the message from the exchange. Second, the extra value g_i depends on the prices of outstanding orders. This is because due to the existence of latency, these outstanding orders may be executed before the market maker can cancel them, and this will affect the future actions and the inventory of the market maker.

One can readily verify from Theorem 2 and (3.1) that for any $\Delta\tau \geq 0$, the market maker's net profit is given by

$$NP = g_0(0,\infty,-\infty). \tag{3.10}$$

The next section is devoted to the analysis of NP.

3.3 Profitability of market making strategies

We now consider the profitability of the market making problem with $\Delta\tau \geq 0$. In particular, we provide explicit conditions under which the market maker can earn positive net profit in the following result.

Theorem 3. *Fix the parameters $\lambda, \lambda^a, \lambda^b, \Delta\tau, \Delta t, \overline{q}, \underline{q}$.*

(1) NP is a non-decreasing function of N.

(2) If $\lambda^a \leq \lambda/2$ and $\lambda^b \leq \lambda/2$, then $NP = 0$ for any $\Delta\tau \geq 0$.

(3) If $\lambda^a > \lambda/2$ and $\lambda^b > \lambda/2$, then there exists a finite positive integer N_{min} depending on the fixed parameters such that

$$NP \begin{cases} = 0, & \textit{for } N < N_{\min}, \\ > 0, & \textit{for } N \geq N_{\min}. \end{cases}$$

We now discuss the implications of Theorem 3. Part (1) of this results suggests that with quoting duration Δt and other parameters fixed, the net profit the market maker can earn is non-decreasing with more quoting opportunities, or equivalently, with a longer trading horizon T as we have $N = \lfloor \frac{T-\Delta\tau}{\Delta t} \rfloor$.

Part (2) of this result says that under the conditions the rates of "uninformed" market orders that transact with market maker's limit orders λ^a, λ^b are smaller than equal to $\lambda/2$, then the market maker earns zero profit. These conditions are likely to hold when (i) the market is highly volatile with a large rate of price change λ, (ii) there are not sufficient "uninformed" market order flows, or (iii) there are sufficient "uninformed" market order flows, but the market maker is not fast enough compared with other participants and hence can not gain good queue positions in the order book. Both (ii) and (iii) lead to low values of λ^a, λ^b. Part (2) then suggests that in these scenarios, electronic market making on the single asset is not profitable, regardless of how low the absolute latency $\Delta\tau$ the market maker experiences (and how many times the market maker quotes). This also illustrates the importance of relative latency for market makers and is consistent with the arms race for speed for high frequency market makers in practice.

Part (3) of this result suggests that the market making strategy can be profitable if the market conditions are good in the sense of $\lambda^a, \lambda^b > \lambda/2$, and market maker can quote a large number of times or have a long trading horizon . This is consistent with the empirical study [26] where they found electronic market makers with longer trading horizons are less susceptible to withdrawing from liquidity provisions. We also remark that one can obtain explicit upper bounds for the threshold N_{min} using model parameters. For details, see the proof of this result in Section B.3 in the appendix.

The comparisons of the rates λ^a, λ^b with $\lambda/2$ in Theorem 3 are closely related to the value of orders we discuss in Section 3.1. In fact, in view of Proposition 1, one can actually show that even with possible latency, we have $\lambda^a \leq \lambda/2$ is equivalent to the value of ask orders $H^{ask}(\Delta\tau, \Delta t, \delta^a) \leq 0$ for any quotes δ^a, and $\lambda^b \leq \lambda/2$ is equivalent to the value of bid orders $H^{ask}(\Delta\tau, \Delta t, \delta^b) \leq 0$ for any quotes δ^b. The proof of Theorem 3 relies on this observation, Theorem 2, and explicit constructions of profitable quoting policies. See Section B.3 for details.

3.4 The effect of latency on market maker's profit

We next present the result on how latency impacts the market maker's net profit.

Proposition 4. *With parameters $\lambda, \lambda^a, \lambda^b, \Delta t, T, \underline{q}, \overline{q}$ fixed, NP is a non-increasing function of latency $\Delta\tau$.*

This proposition shows that for market makers, lower latency leads to higher profits. Figure 5 in Section 4.3 gives a graphical illustration. From the point of view of order values, high latency will increase the chance that the prices of the maker's quotes are crossed by the mid-price which leads to negative order values. This is consistent with [17] which finds that low latency allows market makers to reduce their adverse selection cost. From the point view of inventory risk, high latency will increase the chances of one-sided fills of market maker's bid-ask pair quotes, yielding high inventory cost. For example, if the market price jumps up during the latency period, then the fill probability of maker's ask order increases and that of the bid order decreases, hence the inventory risk may increase. This is consistent with [1] which shows that fast traders can benefit from speed by reducing inventory cost.

4 Numerical experiments

In this section, we present numerical results. Section 4.1 discusses estimations of model parameters using NASDAQ data. In Section 4.2 we present a representative example of the optimal quoting policy of the market maker and the associated inventory process. Section 4.3 is devoted to the analysis of how latency affects the market maker's profit and optimal quoting strategies. Finally, in Section 4.4 we discuss the effect of quoting frequency on the profit. The numerical results are based on the backward recursion in Theorem 2, where we can compute the functions $g_i, i = 0, 1, ..., N$, and find the optimal quotes by truncations of the infinite state and action spaces and using exhaustive search for the maximization problem in (3.8).

4.1 Estimations

We discuss the estimations of model parameters $\lambda, \lambda^a, \lambda^b$ in this section. Other parameters such as the quote duration Δt, the trading horizon T, the inventory bounds $\underline{q}, \overline{q}$, are all chosen by the market maker.

Bid-ask spread	1 tick	2 ticks	≥ 3 ticks
Percentage	88.236	11.757	0.007

Table 1: Percentages of observations with different bid-ask spreads for GE on Oct 3, 2016

4.1.1 Data description

We use NASDAQ's TotalView-ITCH data, which contains message data of order events.[7] The database documents all the order activities causing an update of the limit order book up to the requested number of levels and thus includes visible orders' submissions, cancellations and executions with order reference numbers. Each visible limit order is identified with a unique order reference number which is assigned immediately after the submission. The timestamp of these events is measured in seconds with decimal precision of at least milliseconds and up to nanoseconds depending on the requested number of levels.

We conduct the empirical analysis using one representative stock, General Electric Company (GE), with data from 10:00 a.m. to 4:00 p.m. on a randomly selected day - Oct 3, 2016. Table 1 shows the observations of the bid-ask spreads on that day. As one can see, the spread of GE is 1 tick for the most of the time and the spread is rarely larger than 2 ticks. We also report that the average sizes of limit order queues on best ask and best bid is 9366 and 7811 shares respectively.

4.1.2 Estimations

We first discuss how to estimate λ, the intensity of market price change in our model. As we assume in the model that the bid-ask spread is always 1 tick, we first delete the data when the bid-ask spread is more than 1 tick. Then, we can estimate the intensity of price change using the average number of jumps of mid price per minute during the day. This yields $\lambda = \lambda_{GE} = 1.56$ per minute.

We then discuss how to estimate λ^a, λ^b, the rates of "uninformed" market orders that transacts with the market maker's limit orders at best quotes. It is clear that these rates depend on the speed advantage or "relative latency" of the particular market maker compared with other market participants. Our data does not contain information on who submits orders, so we provide an estimate which provides an upper bound on these rates. To this end, we count a market order as an "uninformed" market order if the mid price does not change after the market order arrives and generates trades, and then we estimate the intensity of arrivals of the total "uninformed" market orders by computing the average numbers of arrivals per minute. For the simplicity in numerical analysis, we use half of the total intensity for buy and sell "uninformed" market orders, which leads to $\lambda^a_{GE} = \lambda^b_{GE} = 1.25$ per minute. Note that λ^a_{GE} and λ^b_{GE} correspond to the case in which the market maker's orders are always on the top of the queue at the best quotes. For a

[7] Data is provided by LOBSTER website (https://lobsterdata.com/).

particular market maker, the rate of " uninformed" market orders (λ^a and λ^b) transacting with his limit orders could be less than λ^a_{GE} and λ^b_{GE}. The extent of the reduction depends on the relative latency or speed advantage of this particular market maker. In the following numerical studies, we will show that this relative latency is very important for the market maker to earn profits. We also remark that it is possible to estimate λ^a and λ^b relying on our theoretical analysis and proprietary trading data of a given market maker. See Appendix C for a short discussion.

4.2 Optimal quotes and inventory processes

Based on the parameters estimated above, we present a representative example of optimal quotes and the associated inventory process in one simulation. See Figure 3.

In this sample path, the market maker quotes every one second in an 100-minute window. The latency is fixed at 0.02 seconds. In total, the market maker sends 4975 ask orders and 5547 bid orders. Among all these orders, 180 ask orders and 179 bid orders are executed. Thus, the order-to-trade ratio, defined as the ratio between the number of orders submitted and that of orders executed, is 29.31. This is typical in high frequency market making where the trader may cancel most of the orders sent and the order-to-trade ratio is usually high.

The left panel of Figure 3 records the optimal quotes. We note that the ask quote with the relative price 14 means there is no ask order sent (due to truncations of the state space in our numerical method). This occurs when the inventory attains the lower bound -4 or when the inventory is -3 and there is an outstanding ask order, see the right panel of Figure 3. It similar for the bid side. We also observe that in this simulation, when the remaining trading time is short, the relative price of the ask quote becomes higher and higher while that of the bid quote does not change. The reason is that the market maker's inventory is negative close to the end of the horizon and the maker needs to unwind it using a market order at the end which is costly. Hence, the market maker sells less aggressively and quotes at the best bid to possibly increase the inventory.

For references, we also plot in Figure 4 the distribution of the profit in 10000 simulations using the same set of parameters. The average profit is 38.21 dollars and the sample standard deviation is 29.94 dollars.

4.3 Effect of Latency on the profit and optimal quotes

In this section we study numerically how latency affects the market maker's profit and optimal quoting policies based on the estimated parameters in Section 4.1.2.

4.3.1 Effect of latency on the net profit

In Figure 5, we show using several representative examples, the market maker's net profit NP as a function of latency $\Delta\tau$ for various λ^a $(= \lambda^b)$. We can make several observations.

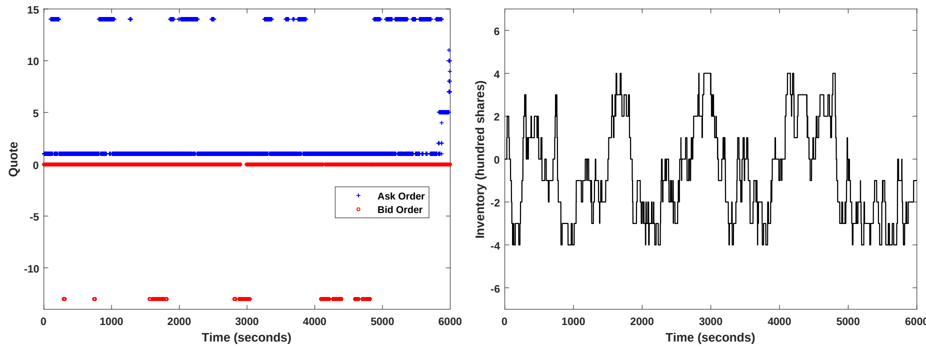

Figure 3: The optimal quotes and the inventory process in the simulation. The model parameters are: $\underline{q} = -4, \overline{q} = 4, \Delta\tau = 0.02$ seconds, $\Delta t = 1$ second, $T = 6000$ seconds, $\lambda = \lambda_{GE}$ and $\lambda^a = \lambda^b = \lambda^a_{GE}$.

First, NP is a non-increasing function of $\Delta\tau$, which is consistent with Proposition 4. In particular, when the latency $\Delta\tau$ is large, then NP becomes zero. This is because the number of quotes $N = 5999$ is fixed in this example, while numerically the threshold $N_{\min}$ for earning positive profit in Theorem 3 increases with $\Delta\tau$. So when the latency is large, we have $N < N_{\min}$ and the net profit of the market maker is zero by Part (3) of Theorem 3.

Second, as indicated by the black line in the figure, when $\lambda^a = \lambda^b = 0.624\lambda^a_{GE} = \frac{\lambda_{GE}}{2}$, NP are always zero for any $\Delta\tau$. This is also consistent with our Theorem 3 in Section 3.

Third, low (absolute) latency is economically important for market makers. For example, consider the case $\lambda^a = \lambda^b = 0.8\lambda_{GE}$. If the market maker can reduce the latency $\Delta\tau$ from 20 millisecond to 10 millisecond, then the market maker's net profit NP will increase from 0.372 to 0.949 dollars by 190% in a 10-minute trading horizon for this single stock.

Finally, relative latency, as indicated by the values of λ^a and λ^b, is also significant for market makers. We find from Figure 5 that for a fixed $\Delta\tau$, NP decreases as λ^a $(= \lambda^b)$ decreases. This is because decreasing λ^a and λ^b will decrease the chance that the orders sent by the maker meet the "uninformed" market orders.

4.3.2 Effect of latency on the optimal quotes

In this section, we illustrate how (absolute) latency affects the optimal action of the market maker for a fixed decision epoch and system state.

Figure 6 shows the optimal quote (δ^a, δ^b) as a function of $\Delta\tau$, where the decision epoch is time zero and the market maker has no initial inventory nor outstanding orders. We can observe that the market maker quotes wider when the latency increases. This is because the market price may move during the latency period, and high latency increases the chances that the prices of the market maker's limit orders are crossed by the mid price, leading to undesirable order executions. Hence, the market maker sends wider quotes to mitigate

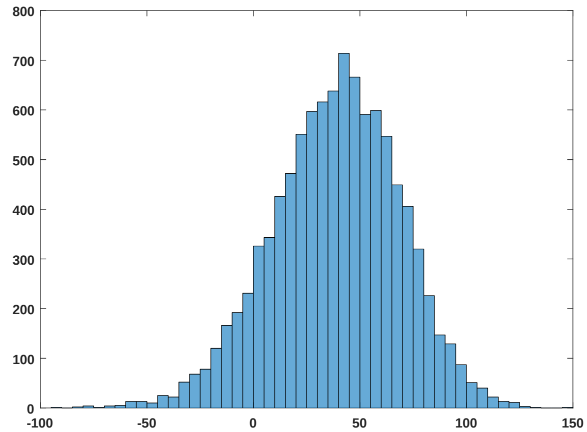

Figure 4: The histogram of the market maker's profit using 10000 simulations. The model parameters are: $\underline{q} = -4, \overline{q} = 4, \Delta\tau = 0.02$ seconds, $\Delta t = 1$ second, $T = 6000$ seconds, $\lambda = \lambda_{GE}$ and $\lambda^a = \lambda^b = \lambda^a_{GE}$.

this increasing risk caused by latency.

We remark that the pattern of optimal quotes in Figure 6 holds for the majority of our extensive experiments with different parameters, system states and decision epochs. However, it does not always hold. When the relative prices of current outstanding orders are asymmetric around mid-price (i.e., $a \neq 1 - b$) and the decision epoch is close to the end of the trading horizon, one can find instances that the optimal quotes do not necessarily become wider when $\Delta\tau$ increases. The reason is that the latency affects not only the quotes sent by the market maker, but also affects the current outstanding orders as the maximum lifetime of the current outstanding orders is $\Delta\tau$.

4.4 Effect of number of quotes on the profit

In this section, we briefly discuss the effect of number of quoting times N on the profit. Corresponding to Theorem 3, we fix all other model parameters and study how NP changes when the quoting times N varies. See Figure 7 for an illustration. Note here the quoting duration $\Delta t = 1$ second is fixed, but the trading horizon $T = N\Delta t + \Delta\tau$ varies. The latency $\Delta\tau = 0.2$ seconds. We can find from Figure 7 that NP is a non-decreasing function N. In addition, when $N > 44$, the profit becomes greater than zero in this example. These two observations are consistent with our Theorem 3. In particular, we can see that enough number of quotes is required for the market maker to make positive profits.

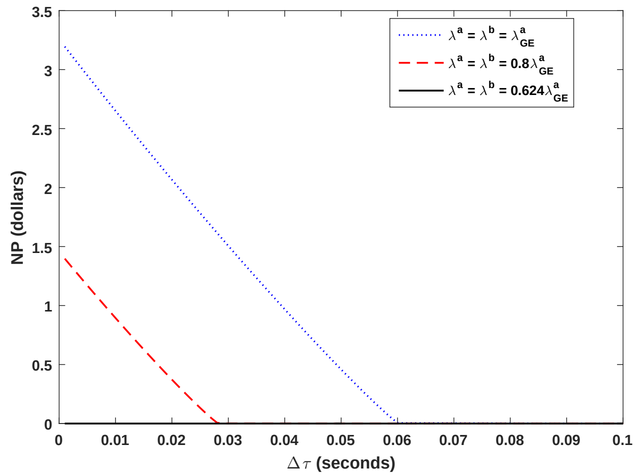

Figure 5: NP as a function of $\Delta\tau$ for various λ^a $(= \lambda^b)$. The remaining model parameters are: $\underline{q} = -2, \overline{q} = 2, \Delta t = 0.1$ seconds, $T = 600$ seconds, and $\lambda = \lambda_{GE}$.

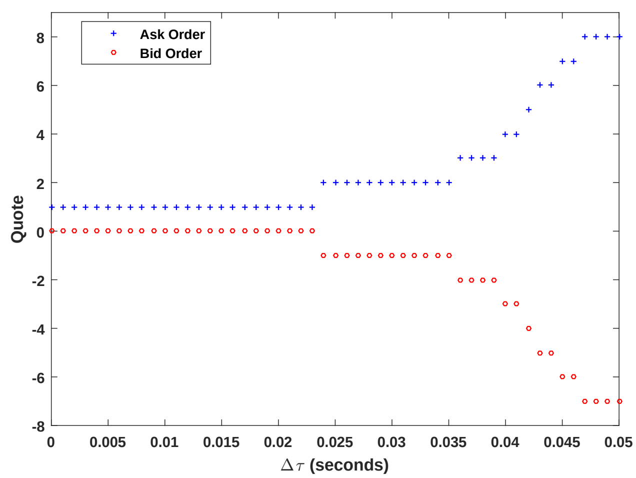

Figure 6: The optimal quote as a function of $\Delta\tau$ at time 0 for the state $s = (w, p, 0, \infty, -\infty)$. The remaining model parameters are: $\underline{q} = -2, \overline{q} = 2, \Delta t = 0.1$ seconds, $T = 120$ seconds, $\lambda = \lambda_{GE}$ and $\lambda^a = \lambda^b = 0.8\lambda^a_{GE}$.

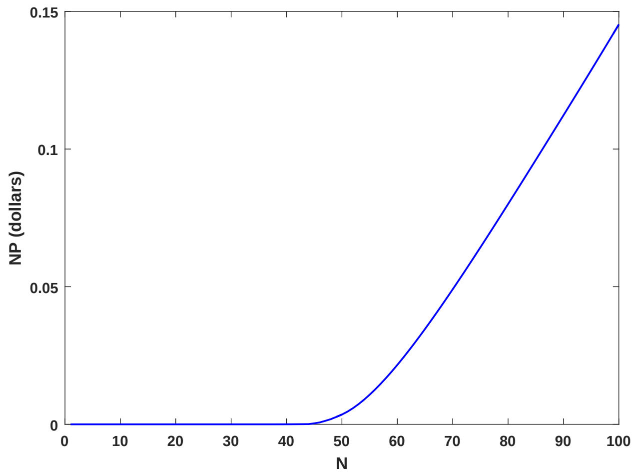

Figure 7: Profit as a function of N (fixed Δt and variable T). The remaining model parameters are $\underline{q} = -2, \overline{q} = 2, \Delta\tau = 0.2$ seconds, $\Delta t = 1$ second, $\lambda = \lambda_{GE}$ and $\lambda^a = \lambda^b = \lambda^a_{GE}$.

5 Conclusion and future research

This work investigates the profitability of electronic market making strategies and the impact of latency on market makers' profits for large–tick assets. By formulating the optimal trading problem in discrete time using Markov decision processes and analyzing the value of orders, we provide an explicit criteria to theoretically determine when an electronic market maker can earn positive profit. We also prove that higher latency leads to reduced profits for market makers, as the asset price may move during the latency period. Numerical experiments are conducted to illustrate the significance of low absolute latency and relative latency for electronic market makers.

A number of simplifying assumptions are made to make our analysis tractable. For example, the market maker studied here is risk neutral in the sense that he maximizes the expected net profit. If the market maker's expected profit is risk-adjusted, then the condition we provide in this paper becomes a necessary condition for such a market maker to earn a positive risk-adjusted profit. In addition, our model captures the relative latency of a market maker in a parsimonious way by directly modeling the "uninformed" market order flows' hitting the market maker's limit orders at best quotes. The rates of these flows generally depend on microstructure information such as recent trades, the state of the order book and the queue positions of the market maker's limit orders. Quantifying this dependence and adding these microstructure features to our model are important extensions that merit further investigation.

A Further details of the Model in Section 2

A.1 Preliminaries

We adopt the following conventions related to the operations for $\pm\infty$: (1) $-\infty < x < \infty$ for any $x \in \mathbb{Z}$; (2) $\infty + x = \infty$ for any $x \in \mathbb{Z} \cup \{\infty\}$; (3) $-\infty + x = -\infty$ for any $x \in \mathbb{Z} \cup \{-\infty\}$; and (4) for any $x \in \mathbb{Z} \cup \{\pm\infty\}$,

$$\pm\infty \times x = \begin{cases} \pm\infty, & x > 0, \\ \mp\infty, & x < 0, \\ 0, & x = 0. \end{cases}$$

A.2 The expression of the admissible action space

In this section we give the expression of the market maker's admissible action space A_s for state $s = (w, p, q, a, b) \in S$. We discuss the zero latency case and positive latency case separately, since the outstanding orders at time t_i will get canceled instantly if latency $\Delta\tau = 0$.

If $\Delta\tau = 0$, the admissible action space for state $s = (w, p, q, a, b) \in S$, denoted as A_s^0, is given by

$$\begin{aligned} A_s^0 = &\{(\delta^a, \delta^b) \in \overline{\mathbb{Z}} \times \overline{\mathbb{Z}} : \\ &\text{if } q = \underline{q}, \text{ then } \delta^a = \infty \text{ or } \delta^b = \infty; \\ &\text{if } q = \overline{q}, \text{ then } \delta^a = -\infty \text{ or } \delta^b = -\infty\}. \end{aligned} \tag{A1}$$

To write down the expression of A_s when latency $\Delta\tau > 0$, we first define two disjoint subsets of the state space S as follows. Write

$$\begin{aligned} \underline{S} := \{(w, p, q, a, b) :&(w, p, q) \in \mathbb{Z}^3, (a, b) \in \overline{\mathbb{Z}} \times \overline{\mathbb{Z}}, \\ &q = \underline{q}, a = \infty, b \leq 0 \\ &\text{or } q = \underline{q} + 1, a < \infty, b \leq 0\}, \end{aligned} \tag{A2}$$

and

$$\begin{aligned} \overline{S} := \{(w, p, q, a, b) :&(w, p, q) \in \mathbb{Z}^3, (a, b) \in \overline{\mathbb{Z}} \times \overline{\mathbb{Z}}, \\ &q = \overline{q}, a \geq 1, b = -\infty \\ &\text{or } q = \overline{q} - 1, a \geq 1, b > -\infty\}. \end{aligned} \tag{A3}$$

The set $\underline{S}$ contains the states in which the market maker's inventory has either reached the lower bound $\underline{q}$ or will reach the lower bound if the outstanding ask order gets filled and the bid order does not get filled. In these cases, the market maker should not quote ask orders or he should use a buy market order in order for the inventory to stay in the bound. The set $\overline{S}$ can be interpreted similarly.

Hence, if latency $\Delta\tau > 0$, the admissible action space for state $s = (w, p, q, a, b) \in S$, denoted as A_s^+, is given by

$$\begin{aligned} A_s^+ = \{ & (\delta^a, \delta^b) \in \overline{\mathbb{Z}} \times \overline{\mathbb{Z}} : \\ & \text{if } s \in \underline{S}, \text{ then } \delta^a = \infty \text{ or } \delta^b = \infty; \\ & \text{if } s \in \overline{S}, \text{ then } \delta^a = -\infty \text{ or } \delta^b = -\infty \}. \end{aligned} \tag{A4}$$

To summarize, we can combine (A1) and (A4) to deduce that the admissible action space for state s is given by

$$A_s = \begin{cases} \{\infty\} \times \overline{\mathbb{Z}} \cup \overline{\mathbb{Z}} \times \{\infty\}, & \Delta\tau = 0, q = \underline{q}, \text{ or } \Delta\tau > 0, s \in \underline{S}, \\ \{-\infty\} \times \overline{\mathbb{Z}} \cup \overline{\mathbb{Z}} \times \{-\infty\}, & \Delta\tau = 0, q = \overline{q}, \text{ or } \Delta\tau > 0, s \in \overline{S}, \\ \overline{\mathbb{Z}}^2, & \text{otherwise.} \end{cases} \tag{A5}$$

A.3 System dynamics for our MDP model

We now describe the dynamics of the discrete system states of the MDP, i.e., s_i, $i = 0, 1, 2, ..., N, N.5$. For $i = 0, 1, ..., N-1$, denote the i-th action/decision of the maker by (δ_i^a, δ_i^b), i.e, the maker sends an (ask,bid) order pair at price $(p_i + \delta_i^a, p_i + \delta_i^b)$ where $p_i = p(t_i-)$ is the market best bid price at time $t_i- = (i \cdot \Delta t)-$. For consistency, we also use the notation δ_N^a, δ_N^b for the last period though there is no decision to make for the maker. Recall that we use $\mathbb{1}_{ask_i}$ and $\mathbb{1}_{bid_i}$ to indicate whether the outstanding ask and bid orders (which exist at t_i- if any) are filled in the time interval $[t_i, t_{i.5})$ respectively; we use $\mathbb{1}_{ask_{i.5}}$ and $\mathbb{1}_{bid_{i.5}}$ to indicate whether the ask and bid orders sent in the i-th action of the maker are filled in the time interval $[t_{i.5}, t_{i+1})$ respectively.

We now describe the dynamics of system states (w, p, q, a, b) from t_i- to $t_{i+1}-$ for $i = 0, 1, .., N-1$. To begin with, we define the price changes $\Delta p_i := p(t_{i.5}) - p(t_i) = \sum_{j=\mathcal{N}(t_i)+1}^{\mathcal{N}(t_{i.5})} X_j$ and $\Delta p_{i.5} := p(t_{i+1}) - p(t_{i.5}) = \sum_{j=\mathcal{N}(t_{i.5})+1}^{\mathcal{N}(t_{i+1})} X_j$. Then we can readily obtain that

$$\begin{aligned} w_{i+1} &= w_i + (p_i + a_i)\mathbb{1}_{ask_i} - (p_i + b_i)\mathbb{1}_{bid_i} \\ &\quad + \max\{p_i + \Delta p_i, p_i + \delta_i^a\}\mathbb{1}_{ask_{i.5}} - \min\{p_i + \Delta p_i + 1, p_i + \delta_i^b\}\mathbb{1}_{bid_{i.5}}, \quad \text{(A6)} \\ p_{i+1} &= p_i + \Delta p_i + \Delta p_{i.5}, \quad \text{(A7)} \\ q_{i+1} &= q_i - \mathbb{1}_{ask_i} + \mathbb{1}_{bid_i} - \mathbb{1}_{ask_{i.5}} + \mathbb{1}_{bid_{i.5}}, \quad \text{(A8)} \\ a_{i+1} &= \mathbb{1}_{ask_{i.5}} \cdot \infty + (1 - \mathbb{1}_{ask_{i.5}})(\delta_i^a - \Delta p_i - \Delta p_{i.5}), \quad \text{(A9)} \\ b_{i+1} &= \mathbb{1}_{bid_{i.5}} \cdot (-\infty) + (1 - \mathbb{1}_{bid_{i.5}})(\delta_i^b - \Delta p_i - \Delta p_{i.5}). \quad \text{(A10)} \end{aligned}$$

The usual conventions for $\pm\infty$ apply, see Section A.1. We briefly explain the dynamics of wealth and outstanding orders as others are straightforward to see. In the wealth dynamics, the market maker earns an amount equal to the execution price if an ask order is filled, and pays an amount equals to the execution price if a bid order is filled. For the outstanding orders, we note that these outstanding orders for the i-th period will be canceled (if not filled) at $t_{i.5} < t_{i+1}$, hence the possible outstanding orders for the $i+1$-th period are from "new" orders sent in the i-th action. Taking the ask side as an example, if such a new ask order is filled in the time interval $[t_{i.5}, t_{i+1})$, i.e., $\mathbb{1}_{ask_{i.5}} = 1$, then at time $t_{i+1}-$, there will be no outstanding ask orders, i.e., $a_{i+1} = \infty$. Otherwise, there will be an outstanding ask order at time $t_{i+1}-$ at price $p_i + \delta_i^a$ and a_{i+1} is given by $p_i + \delta_i^a - p_{i+1} = \delta_i^a - \Delta p_i - \Delta p_{i.5}$, which is always greater than or equal to 1 if $\mathbb{1}_{ask_{i.5}} = 0$. It is similar for the bid side.

We next describe the dynamics from t_N- to $t_{N.5}-$. The dynamics of the market maker' wealth, market best bid price, the maker's inventory, the maker's ask and bid outstanding orders are given as follows:

$$
\begin{aligned}
w_{N.5} &= w_N + (p_N + a_N)\mathbb{1}_{ask_N} - (p_N + b_N)\mathbb{1}_{bid_N}, && \text{(A11)}\\
p_{N.5} &= p_N + \Delta p_N, && \text{(A12)}\\
q_{N.5} &= q - \mathbb{1}_{ask_N} + \mathbb{1}_{bid_N}, && \text{(A13)}\\
a_{N.5} &= \mathbb{1}_{ask_N} \cdot \infty + (1 - \mathbb{1}_{ask_N})(a_N - \Delta p_N), && \text{(A14)}\\
b_{N.5} &= \mathbb{1}_{bid_N} \cdot (-\infty) + (1 - \mathbb{1}_{bid_N})(b_N - \Delta p_N), && \text{(A15)}
\end{aligned}
$$

The main difference compared with the dynamics from t_i to $t_{i+1}, i = 0, 1, .., N-1$ is due to the fact that the market maker only unwinds his inventory position at time t_N without posting new quotes. To see (A14), note that if the ask outstanding order is filled, then there will be no outstanding ask orders at time t_N-. Otherwise, there will be an outstanding ask order at price $p_N + a_N - (p_N + \Delta p_N) = a_N - \Delta p_N$. It is similar for the bid side.

A.4 Formulas of the indicator functions for order executions

In this section, we give the formulas for the indicator functions indicating whether orders are filled. We write $\{\mathcal{N}^a(t) : t \geq 0\}$ and $\{\mathcal{N}^b(t) : t \geq 0\}$ with intensities λ^a and λ^b respectively to denote the "uninformed" buy and sell market order flows that match this particular market maker's limit orders at the best quotes. These two processes are mutually independent and independent with the price process $p(\cdot)$. These two flows model (small) market orders sent by "uninformed" traders such that they do not move price, see, e.g., [10].

We first give a formula for a general definition indicating whether an order is filled. All other indicator functions for order fills we use in this paper are just special cases. Suppose an ask order is sent at time $t \geq 0$ and at relative price $x \in \overline{\mathbb{Z}}$. After a delay $t_1' \geq 0$, it arrives at the order book. We use the indicator function $\mathbb{1}_{ask_{t,t_1',t_2',x}}$ to indicate whether the ask order is filled before time $t + t_1' + t_2'$, i.e, t' time units after it arrives at the exchange.

Assume that none of $\mathcal{N}(t), \mathcal{N}^a(t), \mathcal{N}^b(t)$ jumps at time $t + t'_1$ or $t + t'_1 + t'_2$. To determine whether the ask order is filled, we need to know whether the maximum best bid price in the time interval $[t + t'_1, t + t'_1 + t'_2)$ is larger than the ask order's price $p(t) + x$, called by case 1 (this includes the case that the ask order is filled instantly when it arrives at the order book), or it encounters an "uninformed" buy market order, called by case 2. Thus, for any $0 \le t' \le t''$, we define

$$\begin{aligned}\Delta p^M_{t',t''} :=& \max \quad \{p(t) - p(t') \mid t' \le t \le t''\} \\ =& \max \quad \{ \sum_{j=\mathcal{N}(t')+1}^{n} X_j \mid n = \mathcal{N}(t') + 1, ..., \mathcal{N}(t'')\} \cup \{0\},\end{aligned} \tag{A16}$$

which is the maximum price change in the time interval $[t'_1, t'_2]$. Recall that for any $0 \le t' \le t''$, $\Delta p[t', t''] = p(t'') - p(t')$. Then, case 1 is equivalent to $p(t+t'_1) + \Delta p^M_{t+t'_1, t+t'_1+t'_2} \ge p(t)+x$, i.e., $\Delta p^M_{t+t'_1,t+t'_1+t'_2} \ge x - \Delta p[t, t+t'_1]$. For case 2, for any $0 \le t' \le t''$, we denote by $MI[t', t'']$ the random set of time, when the best bid price attains the maximum best bid price for the time interval $[t', t'']$ as follows:

$$MI[t', t''] := \{t \in [t', t''] \mid p(t) = p(t') + \Delta p^M[t', t'']\}. \tag{A17}$$

For any set of time I, denote by $\mathcal{N}^a(I)$ the number of jumps of $\mathcal{N}^a(t)$ in the time set I. Then, case 2 occurs if the maximum bid price in the time interval $[t + t'_1, t + t'_1 + t'_2]$ is one tick less than the ask order's price, i.e., $\Delta p^M_{t+t'_1,t+t'_1+t'_2} \ge x - \Delta p[t, t+t'_1] - 1$, and there is at least one "uninformed" buy market order that matches the ask order when the ask order stays at the best ask, i.e., $\mathcal{N}^a(MI[t + t'_1, t + t'_1 + t'_2]) \ge 1$. Therefore, the indicate function for this ask order (whether it is filled or not) is given by

$$\mathbb{1}_{ask_{t,t'_1,t'_2,x}} := \mathbb{1}_{\Delta p^M_{t+t'_1,t+t'_1+t'_2} \ge x - \Delta p[t,t+t'_1]} + \mathbb{1}_{\Delta p^M_{t+t'_1,t+t'_1+t'_2} = x - \Delta p[t,t+t'_1]-1} \mathbb{1}_{\mathcal{N}^a(MI[t+t'_1,t+t'_1+t'_2]) \ge 1}. \tag{A18}$$

Then, using the above general definition, we get the indicator functions for ask orders in different cases. For the ask quote sent by the maker at time $t_i, i = 0, 1, ..., N-1$, we have

$$\mathbb{1}_{ask_{i.5}} = \mathbb{1}_{ask_{t_i,\Delta\tau,\Delta t-\Delta\tau,\delta^a_i}}. \tag{A19}$$

Note that an outstanding ask order acts the same as an ask quote sent at the same price without latency. Thus, for the ask outstanding order at time $t_i, i = 0, 1, ..., N$, we have

$$\mathbb{1}_{ask_i} = \mathbb{1}_{ask_{t_i,0,\Delta\tau,a_i}}. \tag{A20}$$

Recall in Equation (3.2), for the definition of the general order value, we have the indicator function $\mathbb{1}_{ask_{t'_1,t'_2,x}}$ for an ask quote sent at time 0. Hence we have

$$\mathbb{1}_{ask_{t'_1,t'_2,x}} = \mathbb{1}_{ask_{0,t'_1,t'_2,x}}. \tag{A21}$$

For the bid side, the formulas of the indicator functions are similar and hence omitted.

B Proofs of results in Section 3

In this section, we give the proofs for the results in Section 3. We first state a lemma on the value of orders. The lemma will be used to establish Theorem 3. In the proof of this lemma, we only need the result in Proposition 1 and the proof of the lemma will be deferred to the end of this section.

Lemma 5. *For any $\Delta\tau \geq 0$, $\Delta t > 0$,*

(a) if $\lambda^a \leq \lambda/2$, then for any $\delta^a \in \overline{\mathbb{Z}}$, $H^{ask}(\Delta\tau, \Delta t, \delta^a) \leq 0$; if $\lambda^b \leq \lambda/2$, then for any $\delta^b \in \overline{\mathbb{Z}}$, $H^{bid}(\Delta\tau, \Delta t, \delta^b) \leq 0$.

(b) if $\lambda^a > \lambda/2$, then there exists $\delta^a \in \mathbb{Z}$, such that $H^{ask}(\Delta\tau, \Delta t, \delta^a) > 0$; if $\lambda^b > \lambda/2$, then there exists $\delta^b \in \mathbb{Z}$, such that $H^{bid}(\Delta\tau, \Delta t, \delta^b) > 0$.

Lemma 5 says if $\lambda^a \leq \lambda/2$, then there are no ask orders whose value is positive and if $\lambda^a > \lambda/2$, there is at least one ask order whose value is positive. It is similar for the bid side. We now present the proofs of our main results.

B.1 Proof of Proposition 1

Proof. We first prove part (a). We only prove it for the ask side. For any $\delta^a \leq 0$, by the definition of indicator functions in Section A.4, we have $\mathbb{1}_{ask_{0,\Delta t,\delta^a}} \equiv 1$. Thus, for any $\delta^a \leq 0$, by Equation (3.2), we have

$$H^{ask}(0, \Delta t, \delta^a) = E[0 - 0.5 - \Delta p[0, \Delta t]] = -0.5.$$

For $\delta^a \geq 1$, we only need to prove that

$$E[\delta^a - \Delta p[0, \Delta t] \mid \mathbb{1}_{ask_{0,\Delta t,\delta^a}} = 1] = \frac{\lambda^a}{\lambda^a + \lambda/2}.$$

Denote the jump times of $\mathcal{N}(t)$ and $\mathcal{N}^a(t)$ by $\tau_1, \tau_2, ...$ and $\tau_1^a, \tau_2^a, ...$ respectively. Define two continuous-time Markov chains (CTMC), $UA(t)$ and $UAsk(t)$ as follows.
For $t \geq 0$,

$$UA(t) := \begin{cases} 1, & \text{if } \{n \in \mathbb{N} : \tau_{\mathcal{N}(t)} < \tau_n^a \leq t\} \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$UAsk(t) := (p(t), UA(t)).$$

$UA(t)$, short for "uninformed" orders at the best ask, indicates if there is any "uninformed" buy market orders ($\mathcal{N}^a(t)$) arrives at the best ask since the last jump time of market price

$\tau_{\mathcal{N}(t)}$. It is easy to see that $Uask(t)$ is a CTMC with state space $\mathbb{Z} \times \{0,1\}$ and the following transition rates:

$$(p,0) \longrightarrow \begin{cases} (p+1,0), & \text{with rate } \lambda/2, \\ (p-1,0), & \text{with rate } \lambda/2, \\ (p,1), & \text{with rate } \lambda^a, \end{cases}$$

and

$$(p,1) \longrightarrow \begin{cases} (p+1,0), & \text{with rate } \lambda/2, \\ (p-1,0), & \text{with rate } \lambda/2, \end{cases}$$

for any $p \in \mathbb{Z}$.

Define the following hitting times:

$$\begin{aligned} \tau_{fill_1} &:= \inf\{t \geq 0 : UAsk(t) = (p(0) + \delta^a - 1, 1)\}, \\ \tau_{fill_2} &:= \inf\{t \geq 0 : UAsk(t) = (p(0) + \delta^a, 0)\}, \\ \tau_{fill} &:= \min\{\tau_{fill_1}, \tau_{fill_2}, \Delta t\}. \end{aligned}$$

If $\tau_{fill_1} < \min\{\tau_{fill_2}, \Delta t\}$, then the ask order sent at time 0 with relative price δ^a and without latency will be filled by an "uninformed" order before time Δt; if $\tau_{fill_2} < \min\{\tau_{fill_1}, \Delta t\}$, then the mid price will cross the price the price of ask order before time Δt. The ask order will be filled before time Δt if and only if $\tau_{fill} < \Delta t$. Using these hitting times, we can decompose $\delta^a - \Delta p[0, \Delta t]$ as follows:

$$\begin{aligned} &E[\delta^a - \Delta p[0, \Delta t] \mid \mathbb{1}_{ask_{0,\Delta t,\delta^a}} = 1] \\ =&E[p(0) + \delta^a - p(\tau_{fill}) \mid \tau_{fill} < \Delta t] - E[p(\Delta t) - p(\tau_{fill}) \mid \tau_{fill} < \Delta t], \end{aligned}$$

where the second term is zero due to optional sampling theorem, noting that the event $\{\tau_{fill} < \Delta t\} \in \mathcal{F}_{\tau_{fill}}$, τ_{fill} and Δt are two bounded stopping times, and $p(t)$ is a martingale. Furthermore, if $\tau_{fill} = \tau_{fill_1}$, then $p(\tau_{fill}) = p(0) + \delta^a - 1$; if $\tau_{fill} = \tau_{fill_2}$, then $p(\tau_{fill}) = p(0) + \delta^a$. Therefore, the fist term of the above formula, which is the quantity we need to prove equal to $\frac{\lambda^a}{\lambda^a + \lambda/2}$, can be represented by the conditional probability of two events related to the hitting times as follows:

$$E[p(0) + \delta^a - p(\tau_{fill}) \mid \tau_{fill} < \Delta t] = \mathbb{P}(\tau_{fill} = \tau_{fill_1} \mid \tau_{fill} < \Delta t).$$

Denote the jump times of $UAsk(t)$ by $\tau_n^{UAsk}, n \geq 1$, then

$$UAsk_n := \begin{cases} (p(0), 0), & n = 0, \\ UAsk(\tau_n^{UAsk}), & n \geq 1, \end{cases}$$

is the embedded discrete-time Markov chain (DTMC) of $UAsk(t)$ with the following transition probability,

$$(p,0) \longrightarrow \begin{cases} (p+1,0), & \text{with probability} \quad \dfrac{\lambda/2}{\lambda^a+\lambda}, \\ (p-1,0), & \text{with probability} \quad \dfrac{\lambda/2}{\lambda^a+\lambda}, \\ (p,1), & \text{with probability} \quad \dfrac{\lambda^a}{\lambda^a+\lambda}, \end{cases}$$

and

$$(p,0) \longrightarrow \begin{cases} (p+1,0), & \text{with probability} \quad 1/2, \\ (p-1,0), & \text{with probability} \quad 1/2. \end{cases}$$

Using the embedded DTMC, we can decompose the two events in the above conditional probability as follows:

$$\{\tau_{fill} = \tau_{fill_1}\} = \bigcup_{n=1}^{\infty} A_n,$$

where

$$\begin{aligned} A_n := &\{UAsk_n = (p(0)+\delta^a-1,1), UAsk_i \notin \{(p(0)+\delta^a-1,1),(p(0)+\delta^a,0)\}, \\ &i=1,2,...,n-1, \sum_i^n \tau_i^{UAsk} < \Delta t\}, \end{aligned}$$

stands for the event that after n transitions, the state of the DTMC first hits $(p(0)+\delta^a-1,1)$ while it never hit the $(p(0)+\delta^a,0)$ before, and moreover the time for the n transitions of the CTMC $UAsk(t)$ is less than Δt (recall that we assume all the Poisson processes do not jump at time Δt). Similarly,

$$\{\tau_{fill} < \Delta t\} = \bigcup_{n=1}^{\infty} B_n,$$

where

$$\begin{aligned} B_n := &\{UAsk_n \in \{(p(0)+\delta^a-1,1),(p(0)+\delta^a,0)\} \\ &UAsk_i \notin \{(p(0)+\delta^a-1,1),(p(0)+\delta^a,0)\}, i=1,2,...,n-1, \sum_i^n \tau_i^{UAsk} < \Delta t\}, \end{aligned}$$

stands for the event that after n transitions, the state of the DTMC first hits $\{(p(0)+\delta^a-1,1),(p(0)+\delta^a,0)\}$ and the time for the n transitions of the CTMC $UAsk(t)$ is less than Δt. Note that $A_n \subseteq B_n, n \geq 1$, and the two series of sets are both pairwise disjoint, i.e., for any $n \neq m$, $A_n \cap A_m = \emptyset$ and $B_n \cap B_m = \emptyset$.

Without loss of any generality, we can assume $p(0) = 0$ for simplicity. Then, we obtain that

$$\begin{aligned}
&\frac{\mathbb{P}(A_n)}{\mathbb{P}(B_n)} = \frac{\mathbb{P}(A_n \cap B_n)}{\mathbb{P}(B_n)} = \mathbb{P}(A_n \mid B_n) \\
=&\mathbb{P}(UAsk_n = (\delta^a - 1, 1), UAsk_i \notin \{(\delta^a - 1, 1), (\delta^a, 0)\}, i = 1, 2, ..., n-1, \sum_i^n \tau_i^{UAsk} < \Delta t \\
&\mid UAsk_n \in \{(\delta^a - 1, 1), (\delta^a, 0)\}, UAsk_i \notin \{(\delta^a - 1, 1), (\delta^a, 0)\}, i = 1, 2, ..., n-1, \\
&\sum_i^n \tau_i^{UAsk} < \Delta t) \\
=&\mathbb{P}(UAsk_n = (\delta^a - 1, 1) \mid UAsk_n \in \{(\delta^a - 1, 1), (\delta^a, 0)\}, UAsk_i \notin \{(\delta^a - 1, 1), (\delta^a, 0)\}, \\
&i = 1, 2, ..., n-1, \sum_i^n \tau_i^{UAsk} < \Delta t) \\
=&\mathbb{P}(UAsk_n = (\delta^a - 1, 1) \\
&\mid UAsk_n \in \{(\delta^a - 1, 1), (\delta^a, 0)\}, UAsk_i \notin \{(\delta^a - 1, 1), (\delta^a, 0)\}, i = 1, 2, ..., n-1) \\
=&\mathbb{P}(UAsk_n = (\delta^a - 1, 1) \mid UAsk_n \in \{(\delta^a - 1, 1), (\delta^a, 0)\}, UAsk_{n-1} = (\delta^a - 1, 0)) \\
=&\frac{\mathbb{P}(UAsk_n = (\delta^a - 1, 1) \mid UAsk_{n-1} = (\delta^a - 1, 0))}{\mathbb{P}(UAsk_n \in \{(\delta^a - 1, 1), (\delta^a, 0)\} \mid UAsk_{n-1} = (\delta^a - 1, 0))} \\
=&\frac{\lambda^a}{\lambda^a + \lambda/2}.
\end{aligned}$$

The forth equality from the end holds because $\{UAsk_i, i = 1, 2, ..., n\}$ and $\{\tau_i^{UAsk}, i = 1, 2, ...\}$ are independent. The third equality from the end holds because of Markov property of $\{UAsk_n\}$ and $UAsk_n \in \{(\delta^a - 1, 1), (\delta^a, 0)\}$ only if $UAsk_{n-1} = (\delta^a - 1, 0)$. Since the ratio between each component pair $(\mathbb{P}(A_n), \mathbb{P}(B_n))$ is the same, we obtain that, for any $\delta^a \geq 1$,

$$\begin{aligned}
&E[\delta^a - \Delta p[0, \Delta t] \mid \mathbb{1}_{ask_{0,\Delta t,\delta^a}} = 1] \\
=&\mathbb{P}(\tau_{fill} = \tau_{fill_1} \mid \tau_{fill} < \Delta t) \\
=&\frac{\sum_{n=1} \mathbb{P}(A_n)}{\sum_{n=1} \mathbb{P}(B_n)} = \frac{\lambda^a}{\lambda^a + \lambda/2}.
\end{aligned}$$

Hence the proof of part (a) is complete for the ask side. The proof for the bid side is similar and hence omitted.

Then, we prove part (b). We prove it for the ask side. By Equation (3.2), we have, for any $\Delta\tau \geq 0$, $\Delta t > 0$ and $\delta^a \in \overline{\mathbb{Z}}$,

$$\begin{aligned}
&H^{ask}(\Delta\tau, \Delta t, \delta^a) \\
=&E[(\max\{\delta^a - \Delta p[0, \Delta\tau], 0\} - 0.5 - \Delta p[\Delta\tau, \Delta\tau + \Delta t])\mathbb{1}_{ask_{\Delta\tau,\Delta t,\delta^a}}] \\
=&\sum_{k=-\infty}^{\infty} E[(\max\{\delta^a - k, 0\} - 0.5 - \Delta p[\Delta\tau, \Delta\tau + \Delta t])\mathbb{1}_{ask_{\Delta\tau,\Delta t,\delta^a}} \\
&\mid \Delta p[0, \Delta\tau] = k]\mathbb{P}(\Delta p[0, \Delta\tau] = k).
\end{aligned}$$

Given that $\Delta p[0, \Delta\tau] = k$, by the formulas of indicator functions in Section A.4, we obtain that

$$\begin{aligned}
\mathbb{1}_{ask_{\Delta\tau,\Delta t,\delta^a}} =&\mathbb{1}_{\Delta p^M_{\Delta\tau,\Delta\tau+\Delta t} \geq \delta^a - \Delta p[0,\Delta\tau]} + \mathbb{1}_{\Delta p^M_{\Delta\tau,\Delta\tau+\Delta t} = \delta^a - \Delta p[0,\Delta\tau] - 1}\mathbb{1}_{\mathcal{N}^a(MI([\Delta\tau,\Delta\tau+\Delta t])) \geq 1} \\
=&\mathbb{1}_{\Delta p^M_{\Delta\tau,\Delta\tau+\Delta t} \geq \delta^a - k} + \mathbb{1}_{\Delta p^M_{\Delta\tau,\Delta\tau+\Delta t} = \delta^a - k - 1}\mathbb{1}_{\mathcal{N}^a(MI([\Delta\tau,\Delta\tau+\Delta t])) \geq 1}.
\end{aligned}$$

Note that $\{(p(t), \mathcal{N}^a(t)) \mid t \geq 0\}$ is a 2-dimensional process with stationary and independent increments. Changing the time interval from $[\Delta\tau, \Delta\tau + \Delta t]$ to $[0, \Delta t]$, we obtain that the following 3-dimensional random vector

$$(\Delta p[\Delta\tau, \Delta\tau + \Delta t], \Delta p^M_{\Delta\tau,\Delta\tau+\Delta t}, \mathcal{N}^a(MI([\Delta\tau, \Delta\tau + \Delta t])))$$

is independent with $\Delta p[0, \Delta\tau]$ and has the same joint distribution as

$$(\Delta p[0, \Delta t], \Delta p^M_{0,\Delta t}, \mathcal{N}^a(MI([0, \Delta t]))).$$

Therefore, for any $k \in \overline{\mathbb{Z}}$, by the definition of $\mathbb{1}_{ask_{0,\Delta t,\delta^a - k}}$, we have

$$\begin{aligned}
&E[(\max\{\delta^a - k, 0\} - 0.5 - \Delta p[\Delta\tau, \Delta\tau + \Delta t])\mathbb{1}_{ask_{\Delta\tau,\Delta t,\delta^a}} \mid \Delta p[0, \Delta\tau] = k] \\
=&E[(\max\{\delta^a - k, 0\} - 0.5 - \Delta p[0, \Delta t])\mathbb{1}_{ask_{0,\Delta t,\delta^a - k}}] \\
=&H^{ask}(0, \Delta t, \delta^a - k).
\end{aligned}$$

Hence,

$$\begin{aligned}
H^{ask}(\Delta\tau, \Delta t, \delta^a - k) =& \sum_{k=-\infty}^{\infty} H^{ask}(0, \Delta t, \delta^a - k)\mathbb{P}(\Delta p[0, \Delta\tau] = k) \\
=&E[H^{ask}(0, \Delta t, \delta^a - \Delta p[0, \Delta\tau])].
\end{aligned}$$

Now the proof of part (b) is complete for the ask side. The proof for the bid side is similar and hence omitted. □

B.2 Proof of Theorem 2

Proof. We prove it by backward induction. For $i = N$, by the Bellman equation, for any $s = (w, p, q, a, b) \in S$,

$$\begin{aligned}
v_N(s) =& E[w_{N.5} + (p_{N.5} + 0.5)q_{N.5} - 0.5|q_{N.5}| \,\big|s_N = s] \\
=& E[w + (p+a)\mathbb{1}_{ask_N} - (p+b)\mathbb{1}_{bid_N} + (p + \Delta p_N + 0.5)(q - \mathbb{1}_{ask_N} + \mathbb{1}_{bid_N}) \\
& - 0.5|q - \mathbb{1}_{ask_N} + \mathbb{1}_{bid_N}| \,\big|s_N = s] \\
=& w + (p+0.5)q + E[(a - 0.5 - \Delta p_N)\mathbb{1}_{ask_N} \mid a_N = a] \\
& + E[(\Delta p_N + 0.5 - b)\mathbb{1}_{bid_N} \mid b_N = b] - 0.5E[\,|q - \mathbb{1}_{ask_N} + \mathbb{1}_{bid_N}| \,\big|(a_N, b_N) = (a, b)] \\
=& w + (p+0.5)q + E[(a - 0.5 - \Delta p_0)\mathbb{1}_{ask_0} \mid a_0 = a] + E[(\Delta p_0 + 0.5 - b)\mathbb{1}_{bid_0} \mid b_0 = b] \\
& - 0.5E[\,|q - \mathbb{1}_{ask_0} + \mathbb{1}_{bid_0}| \,\big|(a_0, b_0) = (a, b)] \\
=& w + (p+0.5)q + H(0, \Delta\tau, a, b) + g_N(q, a, b),
\end{aligned}$$

where the second equality comes from Equations (A11)-(A13), the forth one comes from the stationarity of the MDP, and the fifth one comes from the definitions of functions H, g_N and the indicator functions.

For $i = 0, 1, ..., N-1$, assume $v_{i+1}(s) = w + (p + 0.5)q + H(0, \Delta\tau, a, b) + g_{i+1}(q, a, b)$ for any $s = (w, p, q, a, b) \in S$, then by the Bellman equation, for any $s = (w, p, q, a, b) \in S$,

$$\begin{aligned}
v_i(s) =& \max_{(\delta^a,\delta^b)\in A_s} E[v_{i+1}(s_{i+1}) \mid s_i = s, (\delta_i^a, \delta_i^b) = (\delta^a, \delta^b)] \\
=& \max_{(\delta^a,\delta^b)\in A_s} E[v_{i+1}(s_1) \mid s_0 = s, (\delta_0^a, \delta_0^b) = (\delta^a, \delta^b)] \\
=& \max_{(\delta^a,\delta^b)\in A_s} E[w_1 + (p_1 + 0.5)q_1 + H(0, \Delta\tau, a_1, b_1) + g_{i+1}(q_1, a_1, b_1) \\
& \mid s_0 = s, (\delta_0^a, \delta_0^b) = (\delta^a, \delta^b)] \\
=& \max_{(\delta^a,\delta^b)\in A_s} E[w + (p+a)\mathbb{1}_{ask_0} - (p+b)\mathbb{1}_{bid_0} \\
& + \max\{p + \Delta p_0, p + \delta^a\}\mathbb{1}_{ask_{0.5}} - \min\{p + \Delta p_0 + 1, p + \delta^b\}\mathbb{1}_{bid_{0.5}} \\
& + (p + \Delta p_0 + \Delta p_{0.5} + 0.5)(q - \mathbb{1}_{ask_0} + \mathbb{1}_{bid_0} - \mathbb{1}_{ask_{0.5}} + \mathbb{1}_{bid_{0.5}}) \\
& + H(0, \Delta\tau, a_1, b_1) + g_{i+1}(q_1, a_1, b_1) \mid s_0 = s, (\delta_0^a, \delta_0^b) = (\delta^a, \delta^b)],
\end{aligned}$$

where the second equality comes from the stationarity of the MDP, the third one comes from the assumption for v_{i+1} and the forth one comes from Equations (A6)-(A8).

Reorganizing the terms, we obtain that

$$\begin{aligned}
v_i(s) = & \max_{(\delta^a,\delta^b)\in A_s} \Big\{ w + pq + H^{ask}(0,\Delta\tau,a) - E[\Delta p_{0.5}\mathbb{1}_{ask_0} \mid a_0 = a] + H^{bid}(0,\Delta\tau,b) \\
& + E[\Delta p_{0.5}\mathbb{1}_{bid_0} \mid b_0 = b] + 0.5q + H^{ask}(\Delta\tau,\Delta t - \Delta\tau,\delta^a) + H^{bid}(\Delta\tau,\Delta t-\Delta\tau,\delta^b) \\
& + E[H(0,\Delta\tau,a_1,b_1) + g_{i+1}(q_1,a_1,b_1) \mid (q_0,a_0,b_0) = (q,a,b),(\delta_0^a,\delta_0^b) = (\delta^a,\delta^b)] \Big\} \\
= & w + (p+0.5)q + H(0,\Delta\tau,a,b) - E[\Delta p_{0.5}]E[\mathbb{1}_{ask_0} \mid a_0 = a] \\
& + E[\Delta p_{0.5}]E[\mathbb{1}_{bid_0} \mid b_0 = b] + \max_{(\delta^a,\delta^b)\in A_s} \Big\{ H(\Delta\tau,\Delta t-\Delta\tau,\delta^a,\delta^b) \\
& + E[H(0,\Delta\tau,a_1,b_1) + g_{i+1}(q_1,a_1,b_1) \mid (q_0,a_0,b_0) = (q,a,b),(\delta_0^a,\delta_0^b) = (\delta^a,\delta^b)] \Big\} \\
= & w + (p+0.5)q + H(0,\Delta\tau,a,b) + \max_{(\delta^a,\delta^b)\in A_s} \Big\{ H(\Delta\tau,\Delta t-\Delta\tau,\delta^a,\delta^b) \\
& + E[H(0,\Delta\tau,a_1,b_1) + g_{i+1}(q_1,a_1,b_1) \mid (q_0,a_0,b_0) = (q,a,b),(\delta_0^a,\delta_0^b) = (\delta^a,\delta^b)] \Big\}
\end{aligned}$$

where the first equality comes from the definitions of function H and the indicator functions, the second equality holds because given that $(a_0,b_0) = (a,b)$, $\Delta p_{0.5}$ is independent with $\mathbb{1}_{ask_0}$ and $\mathbb{1}_{bid_0}$, noting that $\mathbb{1}_{ask_0}$ and $\mathbb{1}_{bid_0}$ depend on $\{(\sum_{i=1}^{\mathcal{N}(t)} X_i, \mathcal{N}^a(t), \mathcal{N}^b(t)) : t \in [0,t_{0.5})\}$ while $\Delta p_{0.5} = \sum_{i=\mathcal{N}(t_{0.5})+1}^{\mathcal{N}(t_1)} X_i$, and the third one holds because $\{p(t) \mid t \geq 0\}$ is a martingale.

By the definition of the function G_i, it remains to prove

$$\begin{aligned}
H(\Delta\tau,\Delta t,\delta^a,\delta^b) = & H(\Delta\tau,\Delta t-\Delta\tau,\delta^a,\delta^b) + E[H(0,\Delta\tau,a_1,b_1) \\
& \mid (q_0,a_0,b_0) = (q,a,b),(\delta_0^a,\delta_0^b) = (\delta^a,\delta^b)].
\end{aligned} \tag{B1}$$

Recall that $H(\Delta\tau,\Delta t,\delta^a,\delta^b)$ is the value of a quote pair (δ^a,δ^b) sent by the maker at time 0, in which the mid price for comparison is $p(\Delta\tau+\Delta t)+0.5 = p(t_{1.5})+0.5$, called by value before time $t_{1.5}$; $H(\Delta\tau,\Delta t-\Delta\tau,\delta^a,\delta^b)$ is the value of this quote pair, in which the mid price for comparison is $p(\Delta\tau+\Delta t-\Delta\tau)+0.5 = p(\Delta t)+0.5 = p(t_1)+0.5$, called by value before time t_1; $E[H(0,\Delta\tau,a_1,b_1) \mid (q_0,a_0,b_0) = (q,a,b),(\delta_0^a,\delta_0^b) = (\delta^a,\delta^b)]$ is the expected value of the outstanding orders at time t_1 if any. By the definition of value of quote pairs, Equation (B1) can be divided into the ask part and the bid part. We prove the ask part for by discussing whether the ask order is filled in the time interval $[\tau,t_1)$ or in $[t_1,t_{1.5})$ or not filled. Denote by p_{exe} th execution price of the ask order in this quote pair if the ask order is filled before canceled. First, suppose the ask order is filled before time t_1. Then, the conditional expectation of $p_{exe}-p(t_1)-0.5$ is equal to that of $p_{exe}-p(t_{1.5})-0.5$. This is because the mid price is a martingale with independent increments and is independent with the “uninformed” buy order process $\mathcal{N}^a(t)$ (hence $p(t_{1.5})-p(t_1)$ is independent with the execution of the ask order before time t_1). Meanwhile, there will be no outstanding ask orders at time t_1, hence the expected value of outstanding ask orders at time t_1 is zero. Second, suppose the ask order is filled before in the time interval $[t_1, t_{1.5})$. Then, the value of the ask order before time t_1 is zero since it is not filled before time t_1. Moreover, the execution price of the ask order p_{exe} is the same as that of the outstanding ask order at time t_1 (they are the same order). Thus, the value of the ask order before time $t_{1.5}$ is the same as that of the outstanding ask order at time t_1. Third, suppose the ask order is not filled. Then all the value we mentioned for this order is zero. Therefore, the ask part of Equation (B1) holds. The discussion for the bid part is similar and hence omitted. The proof is thus complete. □

B.3 Proof of Theorem 3

Proof. We first prove part (1). The main idea is given as follows. Comparing two MDP problems with N and $N+1$ periods respectively, the value function at time t_1 in the latter is the same as the value function at t_0 (i.e., the initial one) in the former, because they can be computed by the same backward induction (Bellman equation) from the same terminal value function. For $s = (w, p, 0, \infty, -\infty)$, the value function at t_0 in the $N+1$ period problem is greater than or equal to that at t_1 in the same problem, because the maker can choose to post no orders in the initial action. Thus, NP with $N+1$ periods is greater than or equal to that with N periods.

Mathematically, by Theorem 2 and Equation (3.10), $NP = g_0(0, \infty, -\infty)$ is a function of N. Denote this function by $f_{NP}(N)$. Clearly we have $f_{NP}(0) = 0$. For the two MDP problems with $N = n \geq 1$ and $N = n+1$, denote the value functions and corresponding g function by $v_i^n(s), g_i^n(s), i = 0, 1, ..., n$ and $v_i^{n+1}(s), g_i^{n+1}(s), i = 0, 1, ..., n+1$ respectively. By Theorem 2, functions $v_0^n(s)$ and $v_1^{n+1}(s)$ are given from the same backward induction process starting from the same function

$$v_n^n(s) = v_{n+1}^{n+1}(s) = w + (p+0.5)q + H(0, \Delta\tau, a, b) - 0.5E[\, |q - \mathbb{1}_{ask_0} + \mathbb{1}_{bid_0}| \, \big|(a_0, b_0) = (a, b)],$$

for any $s = (w, p, q, a, b) \in S$. Therefore, for any $s \in S$, $v_0^n(s) = v_1^{n+1}(s)$. By Theorem 2, we have for any $w, p \in \mathbb{Z}$

$$\begin{aligned}
&v_0^{n+1}(w, p, 0, \infty, -\infty) \\
=&w + p * 0 + H(0, \Delta\tau, \infty, -\infty) + \max_{(\delta^a, \delta^b) \in A_s} \Big\{ H(\Delta\tau, \Delta t, \delta^a, \delta^b) + E[g_1^{n+1}(q_1, a_1, b_1) \\
&\mid (q_0, a_0, b_0) = (0, \infty, -\infty), (\delta_0^a, \delta_0^b) = (\delta^a, \delta^b)] \Big\} \\
\geq& w + H(\Delta\tau, \Delta t, \infty, -\infty) + E[g_1^{n+1}(q_1, a_1, b_1) \mid (q_0, a_0, b_0) = (0, \infty, -\infty), (\delta_0^a, \delta_0^b) = (\infty, -\infty)] \\
=&w + g_1^{n+1}(0, \infty, -\infty) = v_1^{n+1}(w, p, 0, \infty, -\infty),
\end{aligned}$$

where the second equality comes from Equation (A8). Thus, we obtain that

$$\begin{aligned} f_{NP}(n+1) &= v_0^{n+1}(w,p,0,\infty,-\infty) - w \\ &\geq v_1^{n+1}(w,p,0,\infty,-\infty) - w = v_0^n(w,p,0,\infty,-\infty) = f_{NP}(n). \end{aligned}$$

Then we prove part (2). By Lemma 5, if $\lambda^a \leq \lambda/2$ and $\lambda^b \leq \lambda/2$, then for any $(\delta^a, \delta^b) \in \overline{\mathbb{Z}}^2$,

$$H(\Delta\tau, \Delta t, \delta^a, \delta^b) = H^{ask}(\Delta\tau, \Delta t, \delta^a) + H^{bid}(\Delta\tau, \Delta t, \delta^b) \leq 0.$$

We prove $g_i(q,a,b) \leq 0$, for $i = 0, 1, ..., N$ and any admissible (q,a,b) by the backward induction in Theorem 2. For $i = N$, it holds directly from Equation (3.8). Suppose for some i $(1 \leq i \leq N)$, $g_i(q,a,b) \leq 0$ for any admissible (q,a,b). Then from Equations (3.8) and (3.9), we obtain that for any admissible (q,a,b).

$$\begin{aligned} g_i(q,a,b) = \max_{(\delta^a,\delta^b)\in A_s} \Big\{ & H(\Delta\tau, \Delta t, \delta^a, \delta^b) + E[g_{i+1}(q_1, a_1, b_1) \\ & \mid (q_0, a_0, b_0) = (q,a,b), (\delta_0^a, \delta_0^b) = (\delta^a, \delta^b)] \Big\} \leq 0. \end{aligned}$$

Therefore, $g_i(q,a,b) \leq 0$, for $i = 0, 1, ..., N$ and any admissible (q,a,b). It follows from Equation (3.10), that

$$NP = g_0(0, \infty, -\infty) \leq 0.$$

Hence, $NP = 0$.

Next, we prove part (3). Suppose $\lambda^a > \lambda/2$ and $\lambda^b > \lambda/2$. We prove for the existence of N_{min}. By Lemma 5, there exists an order pair, denoted by $(\delta_M^a, \delta_M^b) \in \mathbb{Z}^2$, such that $H^{ask}(\Delta\tau, \Delta t, \delta_M^a) > 0$ and $H^{bid}(\Delta\tau, \Delta t, \delta_M^b) > 0$. The main idea is that we construct an admissible policy using this order pair (δ_M^a, δ_M^b), under which the expected profit is positive if N is a sufficiently large even number. Thus, N_{min} exists since $f_{NP}(N)$ is a non-increasing function of N. We also give an upper bound of N_{min} that is an N which is sufficient to make the profit under the admissible policy is positive.

First, we define the admissible policy. Denote by $v_i^\pi(s)$ the expected TW under any admissible policy $\pi = \{f_i : i = 0, 1, ...N\}$ starting at time t_i with initial state $s = (w,p,q,a,b) \in S$. We consider an admissible policy $\tilde{\pi} = \{\tilde{f}_i : i = 0, 1, ...N\}$ in any of our MDP problem with an even $N \geq 4$, i.e., $N = 2K$ for some $2 \leq K \in \mathbb{N}$. Starting at time 0 with an initial state $(w,p,0,\infty,-\infty)$ for any $w, p \in \mathbb{Z}$, $\tilde{\pi}$ is defined as follows. For $i = 1, 3, 5, ..., N-1$, i.e., i is odd, for any $w, p \in \mathbb{Z}, a, b \in \overline{\mathbb{Z}}$, $q \in \{-1, 0, 1\}$ such that $(w,p,q,a,b) \in S$, define

$$\tilde{f}_i(w,p,q,a,b) := (\infty, -\infty).$$

For $i = 2, 4, 6, ..., N-2$, i.e., i is even except 0, for any $w, p \in \mathbb{Z}$, $q \in \{-1, 0, 1\}$, define

$$\tilde{f}_i(w,p,q,\infty,-\infty) := \begin{cases} (\delta_M^a, -\infty), & q = 1, \\ (\infty, -\infty), & q = 0, \\ (\infty, \delta_M^b), & q = -1. \end{cases}$$

For any $w, p \in \mathbb{Z}$, define $\tilde{f}_0(w, p, 0, \infty, \infty) := (\delta^a_M, \delta^b_M)$. Recall that (δ^a_M, δ^b_M) is an action attaining the maximum value of actions. Under this policy, at time $t_i, i = 2, 4, 6, ..., N-2$, the maker will post no orders if his inventory is 0, sell one unit at the relative price δ^a_M if his inventory is 1, and buy one unit at δ^b_M if his inventory is -1. At time $t_i, i = 1, 3, 5, ..., N_1$, the maker does not post any orders. At time t_N, the maker unwinds his inventory if any. At time $t_0 = 0$, there are no inventory or outstanding orders and the maker quotes at (δ^a_M, δ^b_M). Therefore, at time t_i, $i = 0, 2, 4, 6, ..., N-2$, there are no outstanding orders. Moreover, the inventory of the maker always belongs to $\{-1, 0, 1\}$. Hence, the above definition is enough for $\tilde{\pi}$.

Then, we give the backward induction for this policy. Like Bellman equation, standard arguments in MDP theory show that for $i = 2, 4, 6, ..., N-2$, for any $w, p \in \mathbb{Z}$, $q \in \{-1, 0, 1\}$,

$$\begin{aligned} v^{\tilde{\pi}}_i(w, p, q, \infty, -\infty) =& E[v^{\tilde{\pi}}_{i+2}(w_{i+2}, p_{i+2}, q_{i+2}, \infty, -\infty) \mid (w_i, p_i, q_i, a_i, b_i) = (w, p, q, \infty, -\infty), \\ & (\delta^a_i, \delta^b_i) = \tilde{f}_i(w, p, q, \infty, -\infty), (\delta^a_{i+1}, \delta^b_{i+1}) = (\infty, -\infty)], \end{aligned}$$

and

$$\begin{aligned} v^{\tilde{\pi}}_0(w, p, 0, \infty, -\infty) =& E[v^{\tilde{\pi}}_2(w_2, p_2, q_2, \infty, -\infty) \mid (w_0, p_0, q_0, a_0, b_0) = (w, p, 0, \infty, -\infty), \\ & (\delta^a_0, \delta^b_0) = (\delta^a_M, \delta^b_M), (\delta^a_1, \delta^b_1) = (\infty, -\infty)]. \end{aligned}$$

Then, using a similar argument as the proof of Theorem 2, we obtain that, for $i = 2, 4, 6, ..., N$, for any $w, p \in \mathbb{Z}$, $q \in \{-1, 0, 1\}$,

$$\begin{aligned} v^{\tilde{\pi}}_i(w, p, q, \infty, -\infty) =& w + (p + 0.5)q + H(0, \Delta\tau, \infty, -\infty) + g^{\tilde{\pi}}_i(q) \\ =& w + (p + 0.5)q + g^{\tilde{\pi}}_i(q), \end{aligned}$$

where

$$g^{\tilde{\pi}}_N(q) = -0.5|q|,$$

which is because there are no outstanding orders or new quotes sent by the maker at time t_N, and

$$\begin{aligned} g^{\tilde{\pi}}_i(q) =& H(\Delta\tau, \Delta t, \tilde{f}_i(w, p, q, \infty, -\infty)) + E[g^{\tilde{\pi}}_{i+2}(q_2) \mid (q_0, a_0, b_0) = (q, \infty, -\infty), \\ & (\delta^a_0, \delta^b_0) = \tilde{f}_i(w, p, q, \infty, -\infty), (\delta^a_1, \delta^b_1) = (\infty, -\infty)], \text{ for } i = 2, 4, 6, ..., N-2. \end{aligned} \tag{B2}$$

Similarly, for any $w, p \in \mathbb{Z}$, we have

$$\begin{aligned} & v^{\tilde{\pi}}_0(w, p, 0, \infty, -\infty) \\ =& w + (p + 0.5) \cdot 0 + H(0, \Delta\tau, \infty, -\infty) + g^{\tilde{\pi}}_0(0) \\ =& w + g^{\tilde{\pi}}_0(0), \end{aligned}$$

where

$$\begin{aligned} g_0^{\tilde{\pi}}(0) =& H(\Delta\tau, \Delta t, \delta_M^a, \delta_M^b) + E[g_2^{\tilde{\pi}}(q_2) \mid (q_0, a_0, b_0) = (0, \infty, -\infty), \\ & (\delta_0^a, \delta_0^b) = (\delta_M^a, \delta_M^b), (\delta_1^a, \delta_1^b) = (\infty, -\infty)]. \end{aligned} \tag{B3}$$

Next, we prove that if N is sufficiently large, then $g_0^{\tilde{\pi}}(0) > 0$. To do this, by Equation (B3), we only need to prove $g_2^{\tilde{\pi}}(q) \geq 0$ for $q = -1, 0, 1$, since $H(\Delta\tau, \Delta t, \delta_M^a, \delta_M^b) > 0$. First, we prove $g_2^{\tilde{\pi}}(0) = 0$. By Equation (B2), we obtain that, for $i = 2, 4, 6, ..., N-2$,

$$\begin{aligned} g_i^{\tilde{\pi}}(0) =& H(\Delta\tau, \Delta t, \infty, -\infty) + E[g_{i+2}^{\tilde{\pi}}(q_2) \mid (q_0, a_0, b_0) = (0, \infty, -\infty), \\ & (\delta_0^a, \delta_0^b) = (\infty, -\infty), (\delta_1^a, \delta_1^b) = (\infty, -\infty)] \\ =& E[g_{i+2}^{\tilde{\pi}}(0) \mid (q_0, a_0, b_0) = (0, \infty, -\infty), \\ & (\delta_0^a, \delta_0^b) = (\infty, -\infty), (\delta_1^a, \delta_1^b) = (\infty, -\infty)] \\ =& g_{i+2}^{\tilde{\pi}}(0). \end{aligned}$$

Hence, for $i = 2, 4, 6, ..., N-2$, $g_i^{\tilde{\pi}}(0) = g_N^{\tilde{\pi}}(0) = 0$. Then, we prove that if N is sufficiently large, then $g_2^{\tilde{\pi}}(\pm 1) > 0$. Recall that $\mathbb{1}_{ask_{\Delta\tau, \Delta t, \delta_M^a}}$ indicates whether the ask order sent at time 0 and the relative price δ_M^a with latency $\Delta\tau$ is filled in the time interval $[\Delta\tau, \Delta\tau + \Delta t)$. It can be readily verified that when $\delta_0^a = \delta_M^a$ we have

$$\mathbb{1}_{ask_{\Delta\tau, \Delta t, \delta_M^a}} = \mathbb{1}_{ask_{0.5}} + \mathbb{1}_{ask_1}.$$

Intuitively, the ask order is filled if and only if either it is filled in the time interval $[\Delta\tau, \Delta t]$, which is represented by $\mathbb{1}_{ask_{0.5}} = 1$, or it stays as an outstanding order at time $t_1 = \Delta t$, and then filled in the time interval $[\Delta t, \Delta\tau + \Delta t]$, which is represented by $\mathbb{1}_{ask_1} = 1$. Denote the fill probability of this ask order δ_M^a by $p^a := \mathbb{P}(\mathbb{1}_{ask_{\Delta\tau, \Delta t, \delta_M^a}} = 1)$. By Equation (B2), for $i = 2, 4, 6, ..., N-2$, we have

$$\begin{aligned} g_i^{\tilde{\pi}}(1) =& H(\Delta\tau, \Delta t, \delta_M^a, -\infty) + E[g_{i+2}^{\tilde{\pi}}(q_2) \mid (q_0, a_0, b_0) = (1, \infty, -\infty), \\ & (\delta_0^a, \delta_0^b) = (\delta_M^a, -\infty), (\delta_1^a, \delta_1^b) = (\infty, -\infty)] \\ =& H^{ask}(\Delta\tau, \Delta t, \delta_M^a) + E[g_{i+2}^{\tilde{\pi}}(1 - \mathbb{1}_{ask_{0.5}} - \mathbb{1}_{ask_1}) \mid (q_0, a_0, b_0) = (1, \infty, -\infty), \\ & (\delta_0^a, \delta_0^b) = (\delta_M^a, -\infty), (\delta_1^a, \delta_1^b) = (\infty, -\infty)] \\ =& H^{ask}(\Delta\tau, \Delta t, \delta_M^a) + E[g_{i+2}^{\tilde{\pi}}(1 - \mathbb{1}_{ask_{\Delta\tau, \Delta t, \delta_M^a}}) \mid (q_0, a_0, b_0) = (1, \infty, -\infty), \\ & (\delta_0^a, \delta_0^b) = (\delta_M^a, -\infty), (\delta_1^a, \delta_1^b) = (\infty, -\infty)] \\ =& H^{ask}(\Delta\tau, \Delta t, \delta_M^a) + E[g_{i+2}^{\tilde{\pi}}(1 - \mathbb{1}_{ask_{\Delta\tau, \Delta t, \delta_M^a}})] \\ =& H^{ask}(\Delta\tau, \Delta t, \delta_M^a) + p^a g_{i+2}^{\tilde{\pi}}(0) + (1 - p^a) g_{i+2}^{\tilde{\pi}}(1) \\ =& H^{ask}(\Delta\tau, \Delta t, \delta_M^a) + (1 - p^a) g_{i+2}^{\tilde{\pi}}(1), \end{aligned}$$

where the second equality comes from Equation (A8), and the forth equality holds because the random variable $\mathbb{1}_{ask_{\Delta\tau,\Delta t,\delta_M^a}}$ does not depend on $q_0, a_0, b_0, \delta_0^a, \delta_0^b, \delta_1^a$ and δ_1^b. It is similar for the bid side. Define $p^b := \mathbb{P}(\mathbb{1}_{bid_{\Delta\tau,\Delta t,\delta_M^b}} = 1)$. For $i = 2, 4, 6, ..., N-2$, we have

$$g_i^{\tilde{\pi}}(-1) = H^{bid}(\Delta\tau, \Delta t, \delta_M^b) + (1-p^b) g_{i+2}^{\tilde{\pi}}(-1),$$

Recall $N = 2K$. Solving the above recursive equations for $g_i^{\tilde{\pi}}(\pm 1)$ with $g_N^{\tilde{\pi}}(1) = g_N^{\tilde{\pi}}(-1) = -0.5$, we obtain that

$$g_2^{\tilde{\pi}}(1) = (-0.5 - \frac{H^{ask}(\Delta\tau, \Delta t, \delta_M^a)}{p^a})(1-p^a)^{K-1} + \frac{H^{ask}(\Delta\tau, \Delta t, \delta_M^a)}{p^a}, \tag{B4}$$

and

$$g_2^{\tilde{\pi}}(-1) = (-0.5 - \frac{H^{bid}(\Delta\tau, \Delta t, \delta_M^b)}{p^b})(1-p^b)^{K-1} + \frac{H^{bid}(\Delta\tau, \Delta t, \delta_M^b)}{p^b}. \tag{B5}$$

Note that $0 < p^a, p^b < 1$ because $(\delta_M^a, \delta^b) \in \mathbb{Z}^2$. Thus, if N is sufficiently large, then $g_2^{\tilde{\pi}}(\pm 1) > 0$.

Finally, since $\tilde{\pi}$ is an admissible policy, by Equation (3.1), we obtain that

$$NP \geq v_0^{\tilde{\pi}}(w, p, 0, \infty, -\infty) - w = g_0^{\tilde{\pi}}(0) > 0.$$

Note that the backward induction for value functions in each period depends on the model parameters $\lambda, \lambda^a, \lambda^b, \Delta\tau, \Delta t, \overline{q}$ and $\underline{q}$. It follows from the monotonicity of $f_{NP}(N)$ and the fact $f_{NP}(N) > 0$ when N is even and sufficiently large that there exists a constant integer $N_{min} \geq 1$ depending on $\lambda, \lambda^a, \lambda^b, \Delta\tau, \Delta t, \overline{q}$ and $\underline{q}$, such that $f_{NP}(N) > 0$ if and only if $N \geq N_{min}$. For an upper bound of N_{min}, define

$$\bar{N}_{min} := 2 \max \left\{ \left\lceil \frac{\ln \frac{H^{ask}(\Delta\tau,\Delta t,\delta_M^a)}{H^{ask}(\Delta\tau,\Delta t,\delta_M^a)+0.5p_a}}{\ln(1-p_a)} \right\rceil, \left\lceil \frac{\ln \frac{H^{bid}(\Delta\tau,\Delta t,\delta_M^b)}{H^{bid}(\Delta\tau,\Delta t,\delta_M^b)+0.5p_b}}{\ln(1-p_b)} \right\rceil \right\} + 2.$$

It can be readily verified that when $N \geq \bar{N}_{min}$, $g_2^{\tilde{\pi}}(\pm 1) > 0$ and hence $g_0^{\tilde{\pi}}(0) > 0$. Therefore, $\bar{N}_{min} \geq N_{min}$. Now the proof for part (2) is complete. □

B.4 Proof of Proposition 4

Proof. Our purpose is to prove that the net profit of an MDP problem with absolute latency $\Delta\tau_1$ is larger than or equal to that of another problem with latency $\Delta\tau_2$, for any $0 \leq \Delta\tau_1 < \Delta\tau_2 < \Delta t$, while the model parameters $\lambda, \lambda^a, \lambda^b, \Delta t, T, \underline{q}$ and $\overline{q}$ are the same. The main idea of the proof is given as follows. We first modify the two problems to equivalent versions, i.e., the value functions remain unchanged after the modifications. Then we prove that for any admissible policy in the modified problem with latency $\Delta\tau_2$, one can replicate an equivalent policy with the same expected terminal wealth in the modified problem with latency $\Delta\tau_1$. The modifications include change of time intervals, additional decision epochs at which the maker cannot post any orders or cancellation instructions, i.e., he can do nothing, and replacing (non-randomized) Markov policies by history-dependent policies. All these modifications will not change the essence of our MDP problems. In a Markov policy, the decision is a function of the current state (as in our model defined in section 2), while in a history-dependent policy the decision is a function of all states in history (including the current one). We describe these modifications more specifically as follows.

Define the difference of latency $d\tau := \Delta\tau_2 - \Delta\tau_1 > 0$. Define the number of quote pairs $N_1 := \lfloor \frac{T-\Delta\tau_1}{\Delta t} \rfloor$ and $N_2 := \lfloor \frac{T-\Delta\tau_2}{\Delta t} \rfloor$. Clearly, we have either $N_1 = N_2$ or $N_1 = N_2 + 1$. By Theorem 3 part (2), we only need to prove for the case $N_1 = N_2$. Denote the number of quote pairs by $N := N_1 = N_2$. If $N = 0$, then the profit for both latency is zero and the result we need to prove is true. Now suppose $N \geq 1$. Define the action space set $A_{null} := \{act_{null}\}$, where act_{null} stands for posting no orders or cancellation instructions, i.e., doing nothing. We will add some additional decision epochs with this admissible action space. We call those decision epochs *dummy epochs* because they contribute nothing. We call the decision epochs with admissible action spaces defined in Section 2 *real epochs*.

First, we define a problem P_1 with latency $\Delta\tau_1$ as our standard model in Section 2, except that we add an additional time interval $[-d\tau, 0)$ and a dummy epoch at time $-d\tau$. The initial *real epoch* is still at time 0. The underlying continuous-time system state process is denoted by $s^{P_1}(t), -d\tau \leq t \leq N\Delta t + \Delta\tau_1$. $s^{P_1}(t)$ is defined as in Section 2 where the initial time of $p(t), \mathcal{N}^a(t)$ and $\mathcal{N}^b(t)$ is $-d\tau$. Denote the value function at time t (t can be time of any decision epoch), by $v^{P_1,MD}(t,s)$, $s \in S$, where MD standards for Markov deterministic polices.. Due to the stationarity of $p(t), \mathcal{N}^a(t)$ and $\mathcal{N}^b(t)$, $v^{P_1,MD}(0-,s), s \in S$ is the initial value function for an MDP problem with standard definition. Then we define another problem P_1' with latency $\Delta\tau_2$ by modifying P_1 as follows. Define the underlying continuous-time state process $s^{P_1'}(t) := s^{P_1}(t - d\tau)$, $0 \leq t \leq N\Delta t + \Delta\tau_2$. The *real epochs* are defined at times $(i\Delta t + d\tau)-, i = 0, 1..., N-1$, which come from the *real epochs* in problem P_1 by a $d\tau$ time translation. The *dummy epochs* are defined at times $i\Delta t-, i = 0, 1..., N-1, N$, which are the times of decisions in our standard model and $(i\Delta t + d\tau + \Delta\tau_1)- = (i\Delta t + \Delta\tau_2)-, i = 0, 1..., N-1$, which are the arriving times of the maker's order sent in the $i-th$ real epoch. At times $(N\Delta t + d\tau)-$, the maker needs to unwind and the time of terminal state is $(N\Delta t + d\tau + \Delta\tau_1)- = (N\Delta t + \Delta\tau_2)-$. The admissible policy set is either Markovian or history-dependent. Denote the value functions at time t by $v^{P_1',MD}(t,s)$ and $v^{P_1',HD}(t,s), s \in S$ respectively, where HD standards for history-dependent deterministic polices.

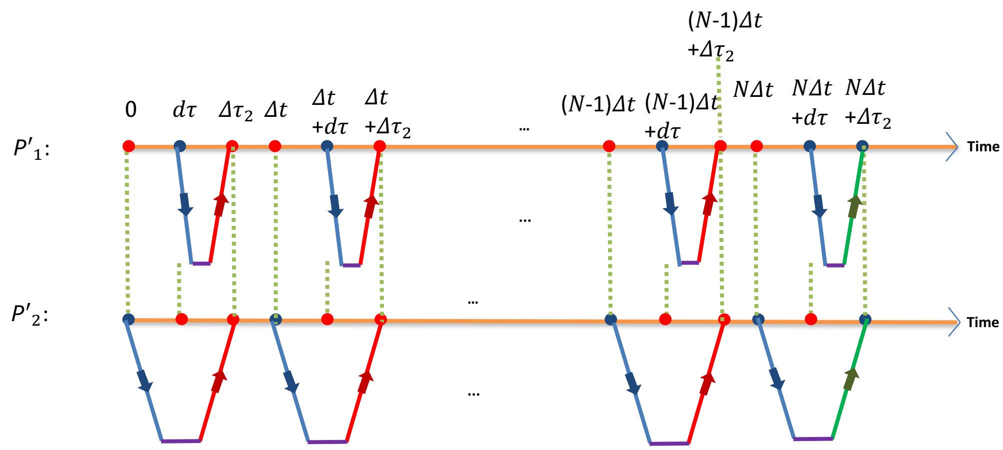

Figure 8: An illustration for proof of Theorem 4

Comparing P_1 and P_1', we obtain that for any $s = (w, p, 0, \infty, -\infty) \in S$,

$$\begin{aligned}
&v^{P_1',HD}(0-, s) = v^{P_1',MD}(0-, s) = v^{P_1,MD}(-d\tau, s)\\
=&E^{P_1,MD}[v^{P_1,MD}(0-, s^{P_1}(0-)) \mid s^{P_1}(-d\tau) = s]\\
=&E^{P_1,MD}[w^{P_1}(0-) + p^{P_1}(0-)q^{P_1}(0-) + g_0^{P_1}(q^{P_1}(0-), a^{P_1}(0-), b^{P_1}(0-)) \mid s^{P_1}(-d\tau) = s]\\
=&E^{P_1,MD}[w + p^{P_1}(0-)0 + g_0^{P_1}(q, \infty, -\infty) \mid s^{P_1}(-d\tau) = s]\\
=&w + g_0^{P_1}(0, \infty, -\infty)\\
=&v^{P_1,MD}(0-, s).
\end{aligned}$$

Here, the first equality comes from Theorem 4.4.1 and 4.4.2 in [25]. The second one holds because of the translation of time and because the *dummy epochs* contribute nothing. The third one comes from the Bellman equation from time $-d\tau$ to $0-$ in problem P_1. The forth one comes from the structure of value function (given in Theorem 2) at time $0-$ in problem P_1, where $(w^{P_1}(t), p^{P_1}(t), q^{P_1}(t), a^{P_1}(t), b^{P_1}(t))$ is system state at time t and $g_0^{P_1}$ is the g_0 function defined in Theorem 2 for problem P_1. The fifth one holds because, in problem P_1, starting without any outstanding orders or inventory and doing nothing at time $-d\tau$, the maker's wealth will remain unchanged during $[-d\tau, 0)$ and there will be no inventory or outstanding order at time $0-$. The sixth one holds because the marker best bid price is a martingale. The seventh one comes from the structure of value function at time $0-$ in problem P_1.

Next we define a standard MDP (as in Section 2) problem P_2 with latency $\Delta\tau_2$. Denote by the underlying continuous-time system state process by $s^{P_2}(t)$, $0 \le t \le N\Delta t + \Delta\tau_2$ and the value function at time t by $v^{P_2,MD}(t, s), s \in S$. Then we define a MDP problem P_2'

with latency $\Delta\tau_2$ by adding some dummy epochs in P_2. We add dummy epochs at times $(i\Delta t + d\tau)-, i = 0, 1..., N-1$, which are times of the real epochs in problem P_1' and times $(i\Delta t + \Delta\tau_2)-, i = 0, 1..., N-1$, which are arriving times of the maker's orders and the same as that in P_1'. The admissible policy set is also either Markovian or history-dependent. Denote the value function at time t by $v^{P_2',MD}(t,s)$ and $v^{P_2',HD}(t,s), s \in S$ respectively. Similarly as in the comparison between P_1 and P_1', we have for any $s = (w, p, 0, \infty, -\infty) \in S$,

$$v^{P_2',HD}(0-, s) = v^{P_2',MD}(0-, s) = v^{P_2,MD}(0-, s).$$

To prove Theorem 4, we need to prove that $v^{P_1,MD}(0-, s) \geq v^{P_2,MD}(0-, s)$, for any $s = (w, p, 0, \infty, -\infty) \in S$. Due to the relationships among the value functions in problems P_1, P_1', P_2 and P_2', we only need to prove $v^{P_1',HD}(0-, s) \geq v^{P_2',HD}(0-, s)$ for any $s = (w, p, 0, \infty, -\infty) \in S$. In problem P_2', for any history-dependent policy and for each *real epoch* at time $i\Delta t, i = 0, 1..., N-1$, the action is a function of the corresponding history states, denoted by $History_i^{P_2'}$. Meanwhile, in problem P_1', for any history-dependent policy, for each *real epoch* at time $i\Delta t + d\tau, i = 0, 1..., N-1$, the action is a function of the corresponding history states, denoted by $History_i^{P_1'}$. Note that $History_i^{P_2'}$ is a subset of $History_i^{P_1'}$, $i = 0, 1, ...N-1$, i.e., the information available at the i-th *real epoch* in P_2' is also available at the i-th *real epoch* in P_1'. This is because the time of the i-th *real epoch* in P_1' is later than that in P_2'. See Figure 8 for an illustration. In Figure 8, red points stand for *dummy epochs* and blue points stand for *real epochs* (and the time to unwind as well as the terminal time). Therefore, for any admissible history-dependent policy in P_2', we can replicate it in P_1' as follows. In P_1', for each *real epoch*, the decision is only based on the information which is available in the corresponding *real epoch* in P_2', and the decision function is the same as the history-dependent policy in P_2'. Note that, in both problems, the orders sent in the i-th *real epoch* arrive the order book at the same time $i\Delta t + \Delta\tau_2$. Therefore, starting with the same initial state, using the replicated policy in P_1', the maker will have the same expected terminal wealth as in P_2'. Thus, we conclude that $v^{P_1',HD}(0, s) \geq v^{P_2',HD}(0, s)$ for any $s = (w, p, 0, \infty, -\infty) \in S$. The proof is complete. $\square$

B.5 Proofs of Lemma 5

This section collects the proof of Lemma 5.

Proof of Lemma 5. We first prove part (a). We prove it for the ask side. Suppose $\lambda^a \leq \lambda/2$. When $\Delta\tau = 0$, by part (a) of Proposition 1, for any $1 \leq \delta^a \in \overline{\mathbb{Z}}$,

$$H^{ask}(0, \Delta t, \delta^a) = \left(\frac{\lambda^a}{\lambda^a + \lambda/2} - 0.5\right) E[\mathbb{1}_{ask_{0.5}} \mid \delta_0^a = \delta^a] \leq 0,$$

and for any $0 \geq \delta^a \in \overline{\mathbb{Z}}$,

$$H^{ask}(0, \Delta t, \delta^a) = -0.5 \leq 0.$$

Thus, when $\tau > 0$, by part (b) of Proposition 1, for any $\delta^a \in \overline{\mathbb{Z}}$,

$$H^{ask}(\tau, \Delta t, \delta^a) = E[H^{ask}(0, \Delta t, \delta^a - \Delta p[0, \tau])] \leq 0.$$

The proof of part (a) is complete for the ask side. The proof for the bid side is similar and hence omitted.

Then we prove part (b). We prove it for the ask side. Suppose $\lambda^a > \lambda/2$. When $\tau = 0$, by part (a) of Proposition 1, when $\delta^a = 1$,

$$H^{ask}(0, \Delta t, 1) = (\frac{\lambda^a}{\lambda^a + \lambda/2} - 0.5)E[\mathbb{1}_{ask_{0.5}} \mid \delta_0^a = 1] > 0.$$

Now suppose $\tau > 0$. By part (b) of Proposition 1, for any $\delta^a \in \mathbb{Z}$, we have

$$\begin{aligned}
&H^{ask}(\tau, \Delta t, \delta^a)\\
=&E[H^{ask}(0, \Delta t, \delta^a - \Delta p[0, \tau])]\\
=&\sum_{k=-\infty}^{\infty} H^{ask}(0, \Delta t, \delta^a - k)\mathbb{P}(\Delta p[0, \tau] = k)\\
=&-0.5\mathbb{P}(\Delta p[0, \tau] \geq \delta^a) + (\frac{\lambda^a}{\lambda^a + \lambda/2} - 0.5)\sum_{k=-\infty}^{\delta^a - 1} E[\mathbb{1}_{ask_{0,\Delta t,\delta^a - k}}]\mathbb{P}(\Delta p[0, \tau] = k)\\
\geq&-0.5\mathbb{P}(\Delta p[0, \tau] \geq \delta^a) + (\frac{\lambda^a}{\lambda^a + \lambda/2} - 0.5)E[\mathbb{1}_{ask_{0,\Delta t,1}}]\mathbb{P}(\Delta p[0, \tau] = \delta^a - 1),
\end{aligned}$$

where the third equality comes from part (a) of Proposition 1. We claim that

$$\lim_{\delta^a \to \infty} \frac{\mathbb{P}(\Delta p[0, \tau] \geq \delta^a)}{\mathbb{P}(\Delta p[0, \tau] = \delta^a - 1)} = 0, \tag{B6}$$

and hence $H^{ask}(\tau, \Delta t, \delta^a) > 0$ if δ^a is sufficiently large.

To prove Equation (B6), we first note that we only need to prove

$$\lim_{\delta^a \to \infty} \frac{\mathbb{P}(\Delta p[0, \tau] = \delta^a)}{\mathbb{P}(\Delta p[0, \tau] = \delta^a - 1)} = 0. \tag{B7}$$

This is because, if Equation (B7) holds, then there exists a constant $c \in (0, 1)$, such that for δ^a sufficiently large, $\frac{\mathbb{P}(\Delta p[0,\tau]=\delta^a)}{\mathbb{P}(\Delta p[0,\tau]=\delta^a-1)} < c$. Thus, as $\delta^a \to \infty$,

$$\frac{\mathbb{P}(\Delta p[0, \tau] \geq \delta^a)}{\mathbb{P}(\Delta p[0, \tau] = \delta^a - 1)} \leq \frac{(1 + c + c^2 + ...)\mathbb{P}(\Delta p[0, \tau] = \delta^a)}{\mathbb{P}(\Delta p[0, \tau] = \delta^a - 1)} = \frac{\frac{1}{1-c}\mathbb{P}(\Delta p[0, \tau] = \delta^a)}{\mathbb{P}(\Delta p[0, \tau] = \delta^a - 1)} \to 0.$$

Next, we prove Equation (B7). For any $1 \leq \delta^a \in \mathbb{Z}$, we have

$$\frac{\mathbb{P}(\Delta p[0,\tau]=\delta^a)}{\mathbb{P}(\Delta p[0,\tau]=\delta^a-1)} = \frac{\sum\limits_{k=\delta^a}^{\infty} \mathbb{P}(\Delta p[0,\tau]=\delta^a \mid \mathcal{N}(\tau)=k)\mathbb{P}(\mathcal{N}(\tau)=k)}{\sum\limits_{k=\delta^a}^{\infty} \mathbb{P}(\Delta p[0,\tau]=\delta^a-1 \mid \mathcal{N}(\tau)=k-1)\mathbb{P}(\mathcal{N}(\tau)=k-1)}, \tag{B8}$$

noting that if $k < \delta^a$, then $\mathbb{P}(\Delta p[0,\tau]=\delta^a \mid \mathcal{N}(\tau)=k)=0$. For any $0 \leq \delta^a \leq k$, because given that $\mathcal{N}(\tau)=k$, $\{p(t)|0 \leq t \leq \tau\}$ is a simple random walk, we have

$$\mathbb{P}(\Delta p[0,\tau]=\delta^a \mid \mathcal{N}(\tau)=k) = \begin{cases} \binom{k}{\frac{k+\delta^a}{2}}\frac{1}{2^k}, & \text{if } k \text{ and } \delta^a \text{ have the same parity,} \\ 0, & \text{if } k \text{ and } \delta^a \text{ have different parities.} \end{cases}$$

Thus, for any $1 \leq \delta^a \leq k$, if δ^a and k have the same parity, then

$$\begin{aligned} &\frac{\mathbb{P}(\Delta p[0,\tau]=\delta^a \mid \mathcal{N}(\tau)=k)\mathbb{P}(\mathcal{N}(\tau)=k)}{\mathbb{P}(\Delta p[0,\tau]=\delta^a-1 \mid \mathcal{N}(\tau)=k-1)\mathbb{P}(\mathcal{N}(\tau)=k-1)} \\ =&\frac{\binom{k}{\frac{k+\delta^a}{2}}\frac{1}{2^k}e^{-\lambda\tau}(\lambda\tau)^k/k!}{\binom{k-1}{\frac{k-1+\delta^a-1}{2}}\frac{1}{2^{k-1}}e^{-\lambda\tau}(\lambda\tau)^{k-1}/(k-1)!} \\ =&\frac{\lambda\tau}{k+\delta^a} \leq \frac{\lambda\tau}{2\delta^a}. \end{aligned}$$

Therefore, by Equation (B8), we obtain that

$$\frac{\mathbb{P}(\Delta p[0,\tau]=\delta^a)}{\mathbb{P}(\Delta p[0,\tau]=\delta^a-1)} \leq \frac{\lambda\tau}{2\delta^a} \to 0,$$

as $\delta^a \to \infty$. Thus, Equation (B7) holds and the proof of part (b) is complete for the ask side. The proof for the bid side is similar and hence omitted. □

C Estimations of λ^a and λ^b

We briefly discuss how a particular market maker can estimate λ^a and λ^b, relying on our theoretical analysis and his own trading data. Taking λ^a as an example as λ^b can be estimated similarly. We first present the following equation, which is derived at the end of this section.

$$\begin{aligned} &E[(1-\Delta p[0,\Delta\tau+\Delta t]-0.5)\mathbb{1}_{ask_{\Delta\tau,\Delta t,1}} \mid \Delta p[0,\Delta\tau]=0] \\ =&(\frac{\lambda^a}{\lambda^a+\lambda/2}-0.5)\cdot E[\mathbb{1}_{ask_{\Delta\tau,\Delta t,1}} \mid \Delta p[0,\Delta\tau]=0]. \end{aligned} \tag{C1}$$

The left hand side of Equation (C1) is the conditional value of the best ask order given that the mid price does not move in the period of latency. The right hand side is a constant multiplied by the conditional fill probability of the best ask order given that the mid price does not move in the period of latency.

The maker can first estimate the conditional value and fill probability mentioned above as well as λ, and then put the estimated quantities into Equation (C1) to compute λ^a. The parameter λ can be estimated by the average number of mid-price jumps. For the conditional value and fill probability, the maker can send multiple best ask orders with the quoting duration Δt (one best ask order every Δt time units). Suppose, there are n ask orders, for which the mid price does not move in the latency period. Among the n orders, suppose there are n_1 orders filled. Then we estimate the conditional fill probability $E[\mathbb{1}_{ask_{\Delta\tau,\Delta t,1}} \mid \Delta p[0,\Delta\tau] = 0]$ by $\frac{n_1}{n}$. Moreover, one can record the differences between the execution prices of the n_1 orders and the mid prices Δt units of time after they enter into the order book. These differences are the realized order value of the n_1 executed orders. Write V_{real} for the sum of these realized order values of the n_1 orders. Note the un-executed $(n - n_1)$ orders are not filled and canceled, which lead to realized order value being zero. Hence, the estimated value of $E[(1 - \Delta p[0, \Delta\tau + \Delta t] - 0.5)\mathbb{1}_{ask_{\Delta\tau,\Delta t,1}} \mid \Delta p[0, \Delta\tau] = 0]$ is $\frac{V_{real}}{n}$. Then, by Equation (C1), we estimate λ^a by $\lambda\frac{n_1+2V_{real}}{2n_1-4V_{real}}$.

Proof of Equation (C1)*.* One can directly compute that

$$
\begin{aligned}
&E[(1 - \Delta p[0, \Delta\tau + \Delta t] - 0.5)\mathbb{1}_{ask_{\Delta\tau,\Delta t,1}} \mid \Delta p[0, \Delta\tau] = 0]\\
=&E[(1 - \Delta p[0, \Delta\tau + \Delta t] - 0.5)(\mathbb{1}_{\Delta p^M_{\Delta\tau,\Delta\tau+\Delta t} \geq 1-\Delta p[0,\tau]}\\
&+ \mathbb{1}_{\Delta p^M_{\Delta\tau,\Delta\tau+\Delta t}=1-\Delta p[0,\Delta\tau]-1}\mathbb{1}_{\mathcal{N}^a(MI[\Delta\tau,\Delta\tau+\Delta t])\geq 1}) \mid \Delta p[0, \Delta\tau] = 0]\\
=&E[(1 - \Delta p[\Delta\tau, \Delta\tau + \Delta t] - 0.5)(\mathbb{1}_{\Delta p^M_{\Delta\tau,\Delta\tau+\Delta t}\geq 1}\\
&+ \mathbb{1}_{\Delta p^M_{\Delta\tau,\Delta\tau+\Delta t}=0}\mathbb{1}_{\mathcal{N}^a(MI[\Delta\tau,\Delta\tau+\Delta t])\geq 1}) \mid \Delta p[0, \Delta\tau] = 0]\\
=&E[(1 - \Delta p[\Delta\tau, \Delta\tau + \Delta t] - 0.5)(\mathbb{1}_{\Delta p^M_{\Delta\tau,\Delta\tau+\Delta t}\geq 1} + \mathbb{1}_{\Delta p^M_{\Delta\tau,\Delta\tau+\Delta t}=0}\mathbb{1}_{\mathcal{N}^a(MI[\Delta\tau,\Delta\tau+\Delta t])\geq 1})]\\
=&E[(1 - \Delta p[0, \Delta t] - 0.5)(\mathbb{1}_{\Delta p^M_{0,\Delta t}\geq 1} + \mathbb{1}_{\Delta p^M_{0,\Delta t}=0}\mathbb{1}_{\mathcal{N}^a(MI[0,\Delta t])\geq 1})]\\
=&E[(1 - \Delta p[0, \Delta t] - 0.5)\mathbb{1}_{ask_{0,\Delta t,1}}]\\
=&\frac{\lambda^a}{\lambda^a + \lambda/2}E[\mathbb{1}_{ask_{0,\Delta t,1}}]\\
=&\frac{\lambda^a}{\lambda^a + \lambda/2}E[\mathbb{1}_{ask_{\Delta\tau,\Delta t,1}} \mid \Delta p[0, \Delta\tau] = 0].
\end{aligned}
$$

Here, the first and fifth equalities are from the formulas of indicator functions in Section A.4, the third equality holds because $p(t)$ has independent increments and is independent with $\mathcal{N}^a(t)$, the forth equality comes from the stationarity of $(p(t), \mathcal{N}^a(t))$, the sixth equality comes from part (b) of Proposition 1, and the last equality can be derived through a similar argument as the first five equalities. □

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