Reinforcement Learning for Jump-Diffusions

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Abstract

We study continuous-time reinforcement learning (RL) for stochastic control in which system dynamics are governed by jump-diffusion processes. We formulate an entropy-regularized exploratory control problem with stochastic policies to capture the exploration–exploitation balance essential for RL. Unlike the pure diffusion case initially studied by Wang et al. (2020), the derivation of the exploratory dynamics under jump-diffusions calls for a careful formulation of the jump part. Through a theoretical analysis, we find that one can simply use the same policy evaluation and q-learning algorithms in Jia and Zhou (2022a, 2023), originally developed for controlled diffusions, without needing to check a priori whether the underlying data come from a pure diffusion or a jump-diffusion. However, we show that the presence of jumps ought to affect parameterizations of actors and critics in general. Finally, we investigate as an application the mean–variance portfolio selection problem with stock price modelled as a jump-diffusion, and show that both RL algorithms and parameterizations are invariant with respect to jumps.

Keywords. Reinforcement learning, continuous time, jump-diffusions, exploratory formulation, well-posedness, Hamiltonian, martingale, q-learning.

1 Introduction

Recently there is an upsurge of interest in continuous-time reinforcement learning (RL) with continuous state spaces and possibly continuous action spaces. Continuous RL problems are important because: 1) many if not most practical problems are naturally continuous in time (and in space), such as autonomous driving, robot navigation, video game play and high frequency trading; 2) while one can discretize time upfront and turn a continuous-time problem into a discrete-time MDP, it has been known, indeed shown experimentally e.g., Munos (2006), Tallec et al. (2019) and Park et al. (2021), that this approach is very sensitive to time discretization and performs poorly with small time steps; 3) there are more analytical tools available for the continuous setting that enable a rigorous and thorough analysis leading to interpretable (instead of black-box) and general (instead of ad hoc) RL algorithms.

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Compared with the vast literature of RL for MDPs, continuous-time RL research is still in its infancy with the latest study focusing on establishing a rigorous mathematical theory and devising resulting RL algorithms. This strand of research starts with Wang et al. (2020) that introduces a mathematical formulation to capture the essence of RL – the exploration–exploitation tradeoff – in the continuous setting, followed by a "trilogy" (Jia and Zhou 2022a,b, 2023) that develops intertwining theories on policy evaluation, policy gradient and q-learning respectively. The common underpinning of the entire theory is the martingale property of certain stochastic processes, the enforcement of which naturally leads to various temporal difference algorithms to train and learn q-functions, value functions and optimal (stochastic) policies. The research is characterized by carrying out all the analysis in the continuous setting, and discretizing time only at the final, implementation stage for approximating the integrated rewards and the temporal difference. The theory has been adapted and extended in different directions; see e.g. Reisinger and Zhang (2021), Guo et al. (2022), Dai et al. (2023), as well as employed for applications; see e.g. Wang and Zhou (2020), Huang et al. (2022), Gao et al. (2022), Wang et al. (2023), and Wu and Li (2024).

The study so far has been predominantly on pure diffusion processes, namely the state processes are governed by controlled stochastic differential equations (SDEs) with a drift part and a diffusion one. While it is reasonable to model the underlying data generating processes as diffusions within a short period of time, sudden and drastic changes can and do happen over time. An example is a stock price process: while it is approximately a diffusion over a sufficiently short period, it may respond dramatically to a surprisingly good or bad earning report. Other examples include neuron dynamics (Giraudo and Sacerdote 1997), stochastic resonance (Gammaitoni et al. 1998) and climate data (Goswami et al. 2018). It is therefore natural and necessary to extend the continuous RL theory and algorithms to the case when jumps are present. This is particularly important for decision makings in financial markets, where it has been well recognized that using jumps to capture large sudden movements provides a more realistic way to model market dynamics; see the discussions in Chapter 1 of Cont and Tankov (2004). The financial modeling literature with jumps dates back to the seminal work of Merton (1976), who extends the classical Black–Scholes model by introducing a compound Poisson process with normally distributed jumps in the log returns. Since then alternative jump size distributions have been proposed in e.g., Kou (2002) and Cai and Kou (2011). Empirical success of jump-diffusion models have been documented for many asset classes; see Bates (1991), Andersen et al. (2002), and Aït-Sahalia et al. (2012) for stocks and stock indices, Bates (1996) for exchange rates, Das (2002) for interest rates, and Li and Linetsky (2014) for commodities, among many others.

This paper makes two major contributions. The first is mathematical in terms of setting up the suitable exploratory formulation and proving the well-posedness of the resulting exploratory SDE, which form the foundation of the RL theory for jump-diffusions. Wang et al. (2020) apply the classical stochastic relaxed control to model the exploration or randomization prevalent in RL, and derive an exploratory state equation that dictates the dynamics of the "average" of infinitely many state processes generated by repeatedly sampling from the same exploratory, stochastic policy. The drift and variance coefficients of the exploratory SDE are the means of those coefficients against the given stochastic policy (which is a probability distribution) respectively. The derivation therein is based on a law of large number argument to the first two moments of the diffusion process. That argument fails for jump-diffusions which are not uniquely determined by the first two moments. We overcome this difficulty by analyzing instead the infinitesimal generator of the sample state process, from which we identify the dynamics of the exploratory state process. Inspired by Kushner (2000) who studies relaxed control for jump-diffusions, we formulate the exploratory SDE by extending the original Poisson random measures for jumps to capture the effect of random exploration. It should be noted that, like almost all the earlier works on relaxed control, Kushner (2000) is motivated by answering the theoretical question of whether an optimal control exists, as randomization convexifies the universe of control strategies. In comparison, our formulation is guided by the practical motivation of exploration for learning. There is also another subtle but important difference. We consider stochastic *feedback* policies while Kushner (2000) does not. This in turn creates technical issues in studying the well-posedness of the exploratory SDE in our framework.

The second main contribution is several implications regarding the impact of jumps on RL algorithm design. Thanks to the established exploratory formulation, we can define the Hamiltonian that, compared with the pure diffusion counterpart, has to include an additional term corresponding to the jumps. The resulting HJB equation – called the exploratory HJB – is now a partial integro-differential equation (PIDE) instead of a PDE due to that additional term. However, when expressed *in terms of the Hamiltonian*, the exploratory HJB equation has exactly the same form as that in the diffusion case. This leads to several completely identical *statements* of important results, including the optimality of the Gibbs exploration, definition of a q-function, and martingale characterizations of value functions and q-functions. Here by "identical" we mean in terms of the Hamiltonian; in other words, these statements differ between diffusions and jump-diffusions entirely because the Hamiltonian is defined differently (which also causes some differences in the proofs of the results concerned). Most important of all, in the resulting RL algorithms, the Hamiltonian (or equivalently the q-function) can be computed using temporal difference of the value function by virtue of the Itô lemma; as a result the algorithms are completely identical no matter whether or not there are jumps. This has a significant practical implication: we can just use the same RL algorithms without the need of checking in advance whether the underlying data come from a pure diffusion or a jump-diffusion. It is significant for the following reason. In practice, data are always observed or sampled at discrete times, no matter how frequent they arrive. Thus we encounter successive discontinuities along the sample trajectory even when the data actually come from a diffusion process. There are some criteria that can be used to check whether the underlying process is a diffusion or a jump-diffusion, e.g. Lehnertz et al. (2018), Wang and Zheng (2022), but they are not always reliable or accurate.

Even though we can apply the same RL algorithms irrespective of the presence of jumps, the *parametrization* of the policy and value function may still depend on it, if we try to exploit certain special structure of the problem instead of using general neural networks for parameterization. Indeed, we give an example in which the optimal exploratory policy is Gaussian when there are no jumps, whereas an optimal policy either does not exist or becomes non-Gaussian when there are jumps. However, in the mean–variance portfolio selection we present as a concrete application, the optimal Gibbs exploration measure again reduces to Gaussian and the value function is quadratic as in Wang and Zhou (2020), both owing to the inherent linear–quadratic (LQ) structure of the problem. Hence in this particular case jumps do not even affect the parametrization of the policy and value function/q-function for learning.

We compare our work with three recent related papers. (1) Bender and Thuan (2023) consider the continuous-time mean–variance portfolio selection problem with exploration under a jump-diffusion setting. Our paper differs from theirs in several aspects. First, they consider a specific application problem while we study RL for general controlled jump-diffusions. Second, they obtain the SDE for the exploratory state by taking limit of the discrete-time exploratory dynamics, whereas our approach first derives the form of the infinitesimal generator of the sample state process and then infers the exploratory SDE from it. It is unclear how their approach works when dealing with general control problems. Finally, they do not consider how to develop RL algorithms based on their solution of the exploratory mean–variance portfolio selection, which we do in this paper. (2) Guo et al. (2023) consider continuous-time RL for linear–convex models with jumps. The scope and motivation are different from ours: They focus on the Lipschitz stability of feedback controls for this special class of control problems where the diffusion and jump terms are not controlled, and propose a least-square model-based algorithm and obtain sublinear regret guarantees in the episodic setting. By contrast, we consider RL for general jump-diffusions and develop model-free algorithms without considering regret bounds. (3) Denkert et al. (2024) aim to unify certain types of stochastic control problems by considering the so-called randomized control formulation which leads to the same optimal value functions as those of the original problems. They develop a policy gradient representation and actor-critic algorithms for RL. The randomized control formulation is fundamentally different from the framework we are considering: therein the control is applied at a set of random time points generated by a random point process instead of at every time point as in our framework.

The remainder of the paper is organized as follows. In Section 2, we discuss the problem formulation. In Section 3, we present the theory of q-learning for jump-diffusions, followed by the discussion of q-learning algorithms in Section 4. In Section 5, we apply the general theory and algorithms to a mean–variance portfolio selection problem. In Section 6, we study the impact of jumps via an example. Finally, Section 7 concludes. All the proofs are given in an appendix.

2 Problem Formulation and Preliminaries

For readers' convenience, we first recall some basic concepts for one-dimensional (1D) Lévy processes, which can be found in standard references such as Sato (1999) and Applebaum (2009). A 1D process $L = \{L_t : t \geq 0\}$ is a Lévy process if it is continuous in probability, has stationary and independent increments, and $L_0 = 0$ almost surely. Denote the jump of L at time t by $\Delta L_t = L_t - L_{t-}$, and let $\mathbf{B}_0$ be the collection of Borel sets of $\mathbb{R}$ whose closure does not contain 0. The Poisson random measure (or jump measure) of L is defined as

$$N(t, B) = \sum_{s:0<s\leq t} 1_B(\Delta L_s), \; B \in \mathbf{B}_0,$$

which gives the number of jumps up to time t with jump size in a Borel set B away from 0. The Lévy measure ν of L is defined by $\nu(B) = \mathbb{E}[N(1, B)]$ for $B \in \mathbf{B}_0$, which shows the expected number of jumps in B in unit time, and $\nu(B)$ is finite. For any $B \in \mathbf{B}_0$, $\{N(t, B) : t \geq 0\}$ is a Poisson process with intensity given by $\nu(B)$. The differential forms of these two measures are written as $N(dt, dz)$ and $\nu(dz)$, respectively. If ν is absolutely continuous, we write $\nu(dz) = \nu(z)dz$ by using the same letter for the measure and its density function. The Lévy measure ν must satisfy the integrability condition $\int_{\mathbb{R}} \min\{z^2, 1\}\nu(dz) < \infty$. However, it is not necessarily a finite measure on $\mathbb{R}/\{0\}$ but always a σ-finite measure. The Lévy process L is said to have jumps of finite (infinite activity) if $\int_{\mathbb{R}} \nu(dz) < \infty$ $(= \infty)$. The number of jumps on any finite time interval is finite in the former case but infinite in the latter. For any Borel set B with its closure including 0, $\nu(B)$ is finite in the finite activity case but infinite otherwise. Finally, the compensated Poisson random measure is defined as $\widetilde{N}(dt,dz) = N(dt,dz) - \nu(dz)dt$. For any $B \in \mathbf{B}_0$, the process $\{\widetilde{N}(t,B) : t \geq 0\}$ is a martingale.

Throughout the paper, we use the following notations. By convention, all vectors are column vectors unless otherwise specified. We use $\mathbb{R}^k$ and $\mathbb{R}^{k\times\ell}$ to denote the space of k-dimensional vectors and $k \times \ell$ matrices, respectively. For matrix A, we use $A^\top$ for its transpose, $|A|$ for its Euclidean/Frobenius norm, and write $A^2 := AA^\top$. Given two matrices A and B of the same size, we denote by $A \circ B$ the inner product between A and B, which is given by $\mathrm{tr}(AB^\top)$. For a positive semidefinite matrix A, we write $\sqrt{A} = UD^{1/2}V^\top$, where $A = UDV^\top$ is its singular value decomposition with U, V as two orthogonal matrices and D as a diagonal matrix, and $D^{1/2}$ is the diagonal matrix whose entries are the square root of those of D. We use $f = f(\cdot)$ to denote the function f, and $f(x)$ to denote the function value of f at x. We use both f_x, f_{xx} and $\frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial x^2}$ for the firs and second (partial) derivatives of a function f with respect to x. We write the minimum of two values a and b as $a \wedge b$. The notation $\mathcal{U}(B)$ denotes the uniform distribution over set B while $\mathcal{N}(\mu, \Sigma)$ refers to the Gaussian distribution with mean vector μ and covariance matrix Σ.

2.1 Classical stochastic control of jump-diffusions

Consider a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ satisfying the usual hypothesis. Assume that this space is rich enough to support $W = \{W_t : t \geq 0\}$, a standard Brownian motion in $\mathbb{R}^m$, and ℓ independent one-dimensional (1D) Lévy processes $L_1, \cdots, L_\ell$, which are also independent of W. Let $N(dt,dz) = (N_1(dt,dz_1), \cdots, N_\ell(dt,dz_\ell))^\top$ be the vector of their Poisson random measures, and similarly define $\nu(dz)$ and $\widetilde{N}(dt,dz)$. The controlled system dynamics are governed by the following Lévy SDE (Øksendal and Sulem 2007, Chapter 3):

$$dX_s^a = b(s, X_{s-}^a, a_s)ds + \sigma(s, X_{s-}^a, a_s)dW_s + \int_{\mathbb{R}^\ell} \gamma(s, X_{s-}^a, a_s, z)\widetilde{N}(ds,dz), \;\; s \in [0,T], \tag{1}$$

where

$$b : [0,T] \times \mathbb{R}^d \times \mathcal{A} \to \mathbb{R}^d, \;\; \sigma : [0,T] \times \mathbb{R}^d \times \mathcal{A} \to \mathbb{R}^{d\times m} \;\; \text{and} \;\; \gamma : [0,T] \times \mathbb{R}^d \times \mathcal{A} \times \mathbb{R}^\ell \to \mathbb{R}^{d\times\ell},$$

a_s is the control or action at time s, $\mathcal{A} \subseteq \mathbb{R}^n$ is the control space, and $a = \{a_s : s \in [0,T]\}$ is the control process assumed to be predictable with respect to $\{\mathcal{F}_s : s \in [0,T]\}$. We denote the k-th column of the matrix γ by γ_k. The goal of stochastic control is, for each initial time-state pair (t,x) of (1), to find the optimal control process a that maximizes the expected total reward:

$$\mathbb{E}\left[\int_t^T e^{-\beta(s-t)} r(s, X_s^a, a_s)ds + e^{-\beta(s-t)} h(X_T^a) \Big| X_t^a = x\right], \tag{2}$$

where $\beta \geq 0$ is a discount factor that measures the time value of the payoff.

The stochastic control problem (1)–(2) is very general; in particular, control processes affect the drift, diffusion and jump coefficients. We now make the following assumption to ensure well-posedness of the problem. Define $\mathbb{R}_K^d := \{x \in \mathbb{R}^d : |x| \leq K\}$.

Assumption 1. *Suppose the following conditions are satisfied by the state dynamics and reward functions:*

(i) b, σ, γ, r, h *are all continuous functions in their respective arguments;*

(ii) (local Lipschitz continuity) for any $K > 0$ *and any* $p \geq 2$*, there exist positive constants* C_K *and* $C_{K,p}$ *such that* $\forall (t,a) \in [0,T] \times \mathcal{A}, (x,x') \in \mathbb{R}^d_K$*,*

$$|b(t,x,a) - b(t,x',a)|^2 + |\sigma(t,x,a) - \sigma(t,x',a)|^2 \leq C_K|x-x'|^2,$$
$$\sum_{k=1}^{\ell} \int_{\mathbb{R}} |\gamma_k(t,x,a,z_k) - \gamma_k(t,x',a,z_k)|^p \nu_k(dz) \leq C_{K,p}|x-x'|^p;$$

(iii) (linear growth in x*) for any* $p \geq 1$*, there exist positive constants* C *and* C_p *such that* $\forall (t,x,a) \in [0,T] \times \mathbb{R}^d \times \mathcal{A}$*,*

$$|b(t,x,a)|^2 + |\sigma(t,x,a)|^2 \leq C(1+|x|^2),$$
$$\sum_{k=1}^{\ell} \int_{\mathbb{R}} |\gamma_k(t,x,a,z)|^p \nu_k(dz) \leq C_p(1+|x|^p);$$

(iv) there exists a constant $C > 0$ *such that*

$$|r(t,x,a)| \leq C\left(1+|x|^p+|a|^q\right),\ |h(x)| \leq C\left(1+|x|^p\right),\ \forall (t,x,a) \in [0,T]\times\mathbb{R}^d\times\mathcal{A}$$

for some $p \geq 2$ *and some* $q \geq 1$*; moreover,* $\mathbb{E}[\int_0^T |a_s|^q ds]$ *is finite.*

Conditions (i)-(iii) guarantee the existence of a unique strong solution to the Lévy SDE (1) with initial condition $X_t^a = x \in \mathbb{R}^d$. Furthermore, for any $p \geq 2$, there exists a constant $C_p > 0$ such that

$$\mathbb{E}_{t,x}\left[\sup_{t\leq s\leq T} |X_s^a|^p\right] \leq C_p(1+|x|^p); \tag{3}$$

see (Kunita 2004, Theorem 3.2) and (Situ 2006, Theorem 119). With the moment estimate (3), it follows that condition (iv) implies that the expected value in (2) is finite.

Let $\mathcal{L}^a$ be the infinitesimal generator associated with the Lévy SDE (1). Under condition (iii), we have $\int_{\mathbb{R}} |\gamma_k(t,x,a,z)|\nu_k(dz) < \infty$ for $k = 1, \cdots, \ell$. Thus, we can write $\mathcal{L}^a$ in the following form:

$$\begin{aligned}\mathcal{L}^a f(t,x) &= \frac{\partial f}{\partial t}(t,x) + b(t,x,a) \circ \frac{\partial f}{\partial x}(t,x) + \frac{1}{2}\sigma^2(t,x,a) \circ \frac{\partial^2 f}{\partial x^2}(t,x) \\ &\quad + \sum_{k=1}^{\ell} \int_{\mathbb{R}} \left(f(t, x+\gamma_k(t,x,a,z)) - f(t,x) - \gamma_k(t,x,a,z) \circ \frac{\partial f}{\partial x}(t,x)\right) \nu_k(dz),\end{aligned} \tag{4}$$

where $\frac{\partial f}{\partial x} \in \mathbb{R}^d$ is the gradient and $\frac{\partial^2 f}{\partial x^2} \in \mathbb{R}^{d\times d}$ is the Hessian matrix.

We recall Itô's formula, which will be frequently used in our analysis; see e.g. (Øksendal and Sulem 2007, Theorem 1.16). Let X^a be the unique strong solution to (1). For any $f \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R}^d)$, we have

$$df(t,X_t^a) = \frac{\partial f}{\partial t}(t,X_{t-}^a)dt + b(t,X_{t-}^a,a_t) \circ \frac{\partial f}{\partial x}(t,X_{t-}^a)dt + \frac{1}{2}\sigma^2(t,X_{t-}^a,a_t) \circ \frac{\partial^2 f}{\partial x^2}(t,X_{t-}^a)dt$$

$$+\sum_{k=1}^{\ell}\int_{\mathbb{R}}\left(f(t,X_{t-}^a+\gamma_k(t,X_{t-}^a,a_t,z))-f(t,X_{t-}^a)-\gamma_k(t,X_{t-}^a,a_t,z)\circ\frac{\partial f}{\partial x}(t,X_{t-}^a)\right)\nu_k(dz)dt$$
$$+\frac{\partial f}{\partial x}(t,X_{t-}^a)\circ\sigma(t,X_{t-}^a,a_t)dW_t+\sum_{k=1}^{\ell}\int_{\mathbb{R}}\left(f(t,X_{t-}^a+\gamma_k(t,X_{t-}^a,a_t,z))-f(t,X_{t-}^a)\right)\widetilde{N}_k(dt,dz).\quad(5)$$

It is known that the Hamilton–Jacobi–Bellman (HJB) equation for the control problem (1)–(2) is given by

$$\sup_{a\in\mathcal{A}}\{r(t,x,a)+\mathcal{L}^aV(t,x)\}-\beta V(t,x)=0,\quad (t,x)\in[0,T)\times\mathbb{R}^d, \tag{6}$$
$$V(T,x)=h(x),$$

where $\mathcal{L}^a$ is given in (4). Under proper conditions, the solution to the above equation is the optimal value function V^* for control problem (2). Moreover, the following function, which maps a time-state pair to an action:

$$a^*(t,x)=\arg\max_{a\in\mathcal{A}}\{r(t,x,a)+\mathcal{L}^aV^*(t,x)\}$$

is the optimal feedback control policy of the problem.

Given a smooth function $V(t,x)\in C^{1,2}([0,T]\times\mathbb{R}^d)$, we define the Hamiltonian H by

$$H(t,x,a,V_x,V_{xx},V)=r(t,x,a)+b(t,x,a)\circ V_x(t,x)+\frac{1}{2}\sigma^2(t,x,a)\circ V_{xx}(t,x)$$
$$+\quad\sum_{k=1}^{\ell}\int_{\mathbb{R}^d}\left(V(t,x+\gamma_k(t,x,a,z))-V(t,x)-\gamma_k(t,x,a,z)\circ V_x(t,x)\right)\nu_k(dz). \tag{7}$$

Then, the HJB equation (6) can be recast as

$$\frac{\partial V(t,x)}{\partial t}+\sup_{a\in\mathcal{A}}\{H(t,x,a,V_x,V_{xx},V)\}-\beta V(t,x)=0,\quad (t,x)\in[0,T)\times\mathbb{R}^d,$$
$$V(T,x)=h(x).$$

2.2 Relaxed control and exploratory formulation

A key idea of RL is to explore the unknown environment by randomizing the actions. Let $\boldsymbol{\pi}:(t,x)\in[0,T]\times\mathbb{R}^d\to\boldsymbol{\pi}(\cdot|t,x)\in\mathcal{P}(\mathcal{A})$ be a given *stochastic* feedback policy, where $\mathcal{P}(\mathcal{A})$ is the set of probability density functions defined on $\mathcal{A}$. Let $\mathbf{a}:(t,x)\in[0,T]\times\mathbb{R}^d\to\mathbf{a}(t,x)\in\mathcal{A}$ be sampled from $\boldsymbol{\pi}$ (i.e. $\mathbf{a}$ is a copy of $\boldsymbol{\pi}$), which is a *deterministic* feedback policy. Applying this policy to (1), we get for $s\in[0,T]$,

$$dX_s^{\mathbf{a}}=b(s,X_{s-}^{\mathbf{a}},\mathbf{a}(s,X_{s-}^{\mathbf{a}}))ds+\sigma(s,X_{s-}^{\mathbf{a}},\mathbf{a}(s,X_{s-}^{\mathbf{a}}))dW_s+\int_{\mathbb{R}^\ell}\gamma(s,X_{s-}^{\mathbf{a}},\mathbf{a}(s,X_{s-}^{\mathbf{a}}),z)\widetilde{N}(ds,dz).$$

Assuming the solution to the above SDE uniquely exists, we say the action process $a^{\boldsymbol{\pi}}=\{a_s^{\boldsymbol{\pi}}=\mathbf{a}(s,X_{s-}^{\mathbf{a}}):t\le s\le T\}$ to be *generated* from $\boldsymbol{\pi}$. Note that $a^{\boldsymbol{\pi}}$ depends on the specific sample $\mathbf{a}\sim\boldsymbol{\pi}$, which we omit to write out for notational simplicity. In the following, we will also write $\boldsymbol{\pi}(\cdot|t,x)$ as $\boldsymbol{\pi}_{t,x}(\cdot)$.

We need to enlarge the original filtered probability space to include the additional randomness from sampling actions. Following Jia and Zhou (2022b, 2023), we assume that the probability space is rich enough to support independent copies of an n-dimensional random vector uniformly distributed over $[0,1]^n$, where n is the dimension of the control space. These copies are also independent of W and $L_1, \cdots, L_\ell$. Let $\mathcal{G}_s$ be the new sigma-algebra generated by $\mathcal{F}_s$ and the copies of the uniform random vector up to time s. The new filtered probability space is $(\Omega, \mathcal{F}, \{\mathcal{G}_t\}, \bar{\mathbb{P}})$, where $\bar{\mathbb{P}}$ is the product extension from $\mathbb{P}$ and they coincide when restricted to $\mathcal{F}_T$.

Fix a stochastic feedback policy $\boldsymbol{\pi}$ and an initial time-state pair (t,x). An action process $a^{\boldsymbol{\pi}} = \{a^{\boldsymbol{\pi}}_s : t \le s \le T\}$ generated from $\boldsymbol{\pi}$ is an $\mathcal{G}_s$-progressively measurable process that is also predictable. Consider the *sample state* process $X^{\boldsymbol{\pi}} = \{X^{\boldsymbol{\pi}}_s : t \le s \le T\}$ that follows the SDE

$$dX^{\boldsymbol{\pi}}_s = b(s, X^{\boldsymbol{\pi}}_{s-}, a^{\boldsymbol{\pi}}_s)ds + \sigma(s, X^{\boldsymbol{\pi}}_{s-}, a^{\boldsymbol{\pi}}_s)dW_s + \int_{\mathbb{R}^\ell} \gamma(s, X^{\boldsymbol{\pi}}_{s-}, a^{\boldsymbol{\pi}}_s, z)\widetilde{N}(ds, dz), \quad s \in [t,T]. \tag{8}$$

Once again, bear in mind that the above equation depends on a specific sample $\mathbf{a} \sim \boldsymbol{\pi}$; so there are in fact infinitely many similar equations, each corresponding to a sample of $\boldsymbol{\pi}$.

To encourage exploration, we add an entropy regularizer to the running reward, leading to

$$J(t,x;\boldsymbol{\pi}) = \mathbb{E}^{\bar{\mathbb{P}}}_{t,x}\left[\int_t^T e^{-\beta(s-t)}\left(r(s, X^{\boldsymbol{\pi}}_s, a^{\boldsymbol{\pi}}_s) - \theta \log \boldsymbol{\pi}(a^{\boldsymbol{\pi}}_s | s, X^{\boldsymbol{\pi}}_{s-})\right) ds + e^{-\beta(T-t)} h(X^{\boldsymbol{\pi}}_T)\right], \tag{9}$$

where $\mathbb{E}^{\bar{\mathbb{P}}}_{t,x}$ is the expectation conditioned on $X^{\boldsymbol{\pi}}_t = x$, taken with respect to the randomness in the Brownian motion, the Poisson random measures, *and* the action randomization. Here $\theta > 0$ is the *temperature parameter* that controls the level of exploration. The function $J(\cdot,\cdot;\boldsymbol{\pi})$ is called the value function of the policy $\boldsymbol{\pi}$. The goal of RL is to find the policy that maximizes the value function among admissible policies that are to be specified in Definition 1 below.

For theoretical analysis, we consider the exploratory dynamics of $X^{\boldsymbol{\pi}}$, which represent the key averaged characteristics of the sample state process over infinitely many randomized actions. In the case of diffusions, Wang et al. (2020) derive such exploratory dynamics by applying a law of large number argument to the first two moments of the diffusion process. Their approach, however, cannot be applied to jump-diffusions. Here, we get around by studying the infinitesimal generator of the sample state process, from which we will identify the dynamics of the exploratory state process.

To this end, let $f \in C^{1,2}_0([0,T) \times \mathbb{R}^d)$, which is continuously differentiable in t and twice continuously differentiable in x with compact support, and we need to analyze $\lim_{s\to 0} \frac{\mathbb{E}^{\bar{\mathbb{P}}}_{t,x}[f(t+s, X^{\boldsymbol{\pi}}_{t+s})] - f(t,x)}{s}$. Fixing (t,x), consider the SDE (8) starting from $X^{\boldsymbol{\pi}}_t = x$ with N independent copies $\mathbf{a}_1, \cdots, \mathbf{a}_N$ of $\boldsymbol{\pi}$. Let $s > 0$ be very small and assume the corresponding actions $a_1, \cdots, a_N$ are fixed from t to $t+s$. Denote by $X^{a_i}_{t+s}$ the value of the state process corresponding to $\mathbf{a}_i$ at $t+s$. Then

$$\begin{aligned}
&\lim_{s\to 0} \frac{\mathbb{E}^{\bar{\mathbb{P}}}_{t,x}[f(t+s, X^{\boldsymbol{\pi}}_{t+s})] - f(t,x)}{s} \\
&= \lim_{s\to 0} \frac{\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^N \mathbb{E}^{\mathbb{P}}_{t,x}[f(t+s, X^{a_i}_{t+s})] - f(t,x)}{s}
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \lim_{s\to 0} \frac{\mathbb{E}^{\mathbb{P}}_{t,x}[f(t+s, X^{a_i}_{t+s})] - f(t,x)}{s} \\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \left(\frac{\partial f}{\partial t}(t,x) + b(t,x,a_i) \circ \frac{\partial f}{\partial x}(t,x) + \frac{1}{2}\sigma^2(t,x,a_i) \circ \frac{\partial^2 f}{\partial x^2}(t,x) \right) \qquad (10) \\
&+ \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \sum_{k=1}^{\ell} \int_{\mathbb{R}} \left(f(t, x+\gamma_k(t,x,a_i,z)) - f(t,x) - \gamma_k(t,x,a_i,z) \circ \frac{\partial f}{\partial x}(t,x) \right) \nu_k(dz). \qquad (11)
\end{aligned}
$$

Using the law of large number, we obtain

$$
(10) = \frac{\partial f}{\partial t}(t,x) + \tilde{b}(t,x,\boldsymbol{\pi}_{t,x}) \circ \frac{\partial f}{\partial x}(t,x) + \frac{1}{2}\tilde{\sigma}^2(t,x,\boldsymbol{\pi}_{t,x}) \circ \frac{\partial^2 f}{\partial x^2}(t,x), \tag{12}
$$

where

$$
\tilde{b}(t,x,\boldsymbol{\pi}_{t,x}) := \int_{\mathcal{A}} b(t,x,a)\boldsymbol{\pi}(a|t,x)da, \quad \tilde{\sigma}(t,x,\boldsymbol{\pi}_{t,x}) := \left(\int_{\mathcal{A}} \sigma^2(t,x,a)\boldsymbol{\pi}(a|t,x)da \right)^{1/2}. \tag{13}
$$

These "exploratory" drift and diffusion coefficients are consistent with those in Wang et al. (2020). It is tempting to think the exploratory jump coefficient $\tilde{\gamma}$ is similarly the average of γ with respect to $\boldsymbol{\pi}$; but unfortunately it is generally not true. This in turn is one of the main distinctive features in studying RL for jump-diffusions.

We approach the problem by analyzing the integrals in (11). Using the second-order Taylor expansion, the boundedness of $\frac{\partial^2 f}{\partial x^2}(t,x)$ for $x \in \mathbb{R}^d$ and condition (iii) of Assumption 1, we obtain that for fixed (t,x) and each k,

$$
\begin{aligned}
&\left| \int_{\mathbb{R}} \left(f(t, x+\gamma_k(t,x,a,z)) - f(t,x) - \gamma_k(t,x,a,z) \circ \frac{\partial f}{\partial x}(t,x) \right) \nu_k(dz) \right| \\
&\leq C \int_{\mathbb{R}} |\gamma_k(t,x,a,z)|^2 \nu_k(dz) \leq C(1+|x|^2)
\end{aligned}
$$

for some constant $C > 0$, which is independent of a. It follows that

$$
(11) = \sum_{k=1}^{\ell} \int_{\mathcal{A}} \int_{\mathbb{R}} \left(f(t, x+\gamma_k(t,x,a,z)) - f(t,x) - \gamma_k(t,x,a,z) \circ \frac{\partial f}{\partial x}(t,x) \right) \nu_k(dz)\boldsymbol{\pi}(a|t,x)da. \tag{14}
$$

Combining (12) and (14), the infinitesimal generator $\mathcal{L}^{\boldsymbol{\pi}}$ of the sample state process is given by the probability weighted average of the generator $\mathcal{L}^a$ of the classical controlled jump-diffusion, i.e.,

$$
\mathcal{L}^{\boldsymbol{\pi}} f(t,x) = \int_{\mathcal{A}} \mathcal{L}^a f(t,x)\boldsymbol{\pi}(a|t,x)da. \tag{15}
$$

Next, we reformulate the integrals in (14) to convert them to the same form as (4), from which we can infer the SDE for the exploratory state process.

Recall that the Poisson random measure $N_k(dt,dz)$ with intensity measure $dt\nu_k(dz)$ ($k = 1,\cdots,\ell$) is defined over the product space $[0,T] \times \mathbb{R}$. We can also interpret N_k as a counting measure associated with a random configuration of points $(T_i, Y_i) \in [0,T] \times \mathbb{R}$ (Cont and Tankov 2004, Section 2.6.3), i.e.,

$$N_k = \sum_{i \geq 1} \delta_{(T_i, Y_i)},$$

where δ_x is the Dirac measure with mass one at point x, T_i is the arrival time of the ith jump of the Lévy process L_k, and $Y_i = L_k(T_i) - L_k(T_i-)$ is the size of this jump. We can interpret Y_i as the mark of the ith event.

At T_i, the size of the jump in the controlled state X under policy $\boldsymbol{\pi}$ is given by $\gamma(T_i, X^{\boldsymbol{\pi}}_{T_i-}, a^{\boldsymbol{\pi}}_{T_i}, Y_i)$, where $X^{\boldsymbol{\pi}}_{T_i-}$ is the state right before the jump occurs and $a^{\boldsymbol{\pi}}_{T_i}$ is the action generated from the feedback policy $\boldsymbol{\pi}$. When the policy $\boldsymbol{\pi}$ is deterministic, the generated action is determined by $(T_i, X^{\boldsymbol{\pi}}_{T_i-})$ and thus the size of the jump in $X^{\boldsymbol{\pi}}$ is a function of $(T_i, X^{\boldsymbol{\pi}}_{T_i-}, Y_i)$. By contrast, when $\boldsymbol{\pi}$ becomes stochastic, an additional random noise is introduced at T_i that determines the generated action together with $(T_i, X^{\boldsymbol{\pi}}_{T_i-})$. Consequently, the size of the jump in $X^{\boldsymbol{\pi}}$ is a function of $(T_i, X^{\boldsymbol{\pi}}_{T_i-}, Y_i)$ plus the random noise for exploration at T_i.

This motivates us to construct new Poisson random measures on an extended space to capture the effect of random noise on jumps for stochastic policies. Specifically, for each $k = 1, \cdots, \ell$, we construct a new Poisson random measure, denoted by $N'_k(dt, dz, du)$, on the product space $[0,T] \times \mathbb{R} \times [0,1]^n$, with its intensity measure given by $dt\nu_k(dz)du$. Here, u is the realized value of the n-dimensional random vector U that follows $\mathcal{U}([0,1]^n)$, which is the random noise introduced in the probability space for exploration. The new Poisson random measure N'_k is also a counting measure associated with a random configuration of points $(T_i, Y_i, U_i) \in [0,T] \times \mathbb{R} \times [0,1]^n$:

$$N'_k = \sum_{i \geq 1} \delta_{(T_i, Y_i, U_i)},$$

where T_i and Y_i are the same as above, and U_i is the $\mathcal{U}([0,1]^n)$ random vector that generates random exploration at T_i. Hence, the ith event is marked by both Y_i and U_i under N'_k. We let $N'(dt, dz, du) = (N'_1(dt, dz_1, du), \cdots, N'_\ell(dt, dz_\ell, du))^\top$.

In general, for any n-dimensional random vector ξ that follows distribution $\boldsymbol{\pi}$, we can find a measurable function $G^{\boldsymbol{\pi}} : \mathbb{R}^n \to \mathbb{R}^n$ such that $\xi = G^{\boldsymbol{\pi}}(U)$, where $U \sim \mathcal{U}([0,1]^n)$. As an example, consider $\xi \sim \mathcal{N}(\mu, AA^\top)$. We can represent it as $\xi = \mu + A\Phi^{-1}(U)$, where Φ is the cumulative distribution function of the univariate standard normal distribution and $\Phi^{-1}(U)$ is a vector obtained by applying Φ^{-1} to each component of U.

For the stochastic feedback policy $\boldsymbol{\pi}_{t,x}$, using $a = G_{\boldsymbol{\pi}_{t,x}}(u)$ we obtain

$$\begin{aligned}
&\int_{\mathcal{A}} \int_{\mathbb{R}} \left(f(t, x + \gamma_k(t,x,a,z)) - f(t,x) - \gamma_k(t,x,a,z) \circ \frac{\partial f}{\partial x}(t,x) \right) \nu_k(dz) \boldsymbol{\pi}(a|t,x) da \\
&= \int_{\mathbb{R} \times [0,1]^n} \left(f\left(t, x + \gamma_k\left(t, x, G^{\boldsymbol{\pi}_{t,x}}(u), z\right)\right) - f(t,x) - \gamma_k\left(t, x, G^{\boldsymbol{\pi}_{t,x}}(u), z\right) \circ \frac{\partial f}{\partial x}(t,x) \right) \nu_k(dz) du.
\end{aligned}$$

It follows that the infinitesimal generator of the sample state process can be written as

$$\begin{aligned}
&\mathcal{L}^{\boldsymbol{\pi}} f(t,x) = \frac{\partial f}{\partial t}(t,x) + \tilde{b}(s, x, \boldsymbol{\pi}_{t,x}) \circ \frac{\partial f}{\partial x}(t,x) + \frac{1}{2} \tilde{\sigma}^2(s, x, \boldsymbol{\pi}_{t,x}) \circ \frac{\partial^2 f}{\partial x^2}(t,x) \\
&+ \sum_{k=1}^{\ell} \int_{\mathbb{R} \times [0,1]^n} \left(f\left(t, x + \gamma_k\left(t, x, G^{\boldsymbol{\pi}_{t,x}}(u), z\right)\right) - f(t,x) - \gamma_k\left(t, x, G^{\boldsymbol{\pi}_{t,x}}(u), z\right) \circ \frac{\partial f}{\partial x}(t,x) \right) \nu_k(dz) du.
\end{aligned} \tag{16}$$

Comparing (16) with (4) in terms of the integral part that characterizes the behavior of jumps, we observe that the new measure $\nu_k(dz)du$ replaces the Lévy measure $\nu_k(dz)$ and integration is done over an extended space to capture the effect of random exploration on jumps. The jump coefficient function that generates the jump size in the controlled state process X given the Lévy jump with size z and control variable a is still the same. However, in (16) the control a is generated from u as $G^{\boldsymbol{\pi}_{t,x}}(u)$, where u is the realized value of the random noise introduced for exploration. In the following, we will also write $G^{\boldsymbol{\pi}_{t,x}}(u)$ as $G^{\boldsymbol{\pi}}(t,x,u)$ whenever using the latter simplifies notations.

Based on (16), we see that the exploratory state should be the solution to the following Lévy SDE:

$$\begin{aligned} dX_s^{\boldsymbol{\pi}} &= \tilde{b}(s, X_{s-}^{\boldsymbol{\pi}}, \boldsymbol{\pi}(\cdot|s, X_{s-}^{\boldsymbol{\pi}}))ds + \tilde{\sigma}(s, X_{s-}^{\boldsymbol{\pi}}, \boldsymbol{\pi}(\cdot|s, X_{s-}^{\boldsymbol{\pi}}))dW_s \\ &\quad + \int_{\mathbb{R}\times[0,1]^n} \gamma\left(s, X_{s-}^{\boldsymbol{\pi}}, G^{\boldsymbol{\pi}}(s, X_{s-}^{\boldsymbol{\pi}}, u), z\right) \widetilde{N}'(ds, dz, du),\ X_t^{\boldsymbol{\pi}} = x,\ s \in [t, T], \end{aligned} \tag{17}$$

which we call the *exploratory Lévy SDE*. The solution process, if exists, is denoted by $\tilde{X}^{\boldsymbol{\pi}}$ and called the exploratory (state) process. As we explain below, this process informs us the behavior of the key characteristics of the sample state process after averaging over infinitely many actions sampled from the stochastic policy $\boldsymbol{\pi}$.

In general, the sample state process $X^{\boldsymbol{\pi}}$ defined by (8) is a semimartingale, as it is the sum of three processes: the drift process that has finite variation (the first term in (8)), the continuous (local) martingale driven by the Brownian motion (the second term in (8)), and the discontinuous (local) martingale driven by the compensated Poisson random measure (the third term in (8)). Any semimartingale is fully determined by three characteristics: the drift, the quadratic variation of the continuous local martingale, and the compensator of the random measure associated with the process's jumps (the compensator gives the jump intensity); see Jacod and Shiryaev (2013) for detailed discussions of semimartingales and their characteristics.

For the sample state process, given that $X_{s-}^{\boldsymbol{\pi}} = x$ and the action sampled from $\boldsymbol{\pi}_{s,x}$ is $a \in \mathcal{A}$, the characteristics over an infinitesimally small time interval $[s, s+ds]$ are given by the triplet $(b(s,x,a)ds, \sigma^2(s,x,a)ds, \sum_{k=1}^{\ell} \gamma_k(s,x,a,z)\nu_k(dz)ds)$.

Now consider the exploratory state process $\tilde{X}^{\boldsymbol{\pi}}$, which is also a semimartingale by (17). Its characteristics over an infinitesimally small time interval $[s, s+ds]$ with $\tilde{X}_{s-}^{\boldsymbol{\pi}} = x$ are given by the triplet $(\tilde{b}(s,x,\boldsymbol{\pi}_{s,x})ds, \tilde{\sigma}^2(s,x,\boldsymbol{\pi}_{s,x})ds, \sum_{k=1}^{\ell}\int_{[0,1]^n}\gamma_k(s,x,G(s,x,u),z)du\cdot\nu_k(dz)ds)$, where the third characteristic is obtained by calculating $\mathbb{E}\left[\sum_{k=1}^{\ell}\gamma_k\left(s,x,G^{\boldsymbol{\pi}}(s,x,u),z\right)N_k(ds,dz,du)\right]$ for Lévy jumps with size from $[z, z+dz]$. Using (13), we have

$$\tilde{b}(s,x,\boldsymbol{\pi}_{s,x})ds = \int_{\mathcal{A}} b(s,x,a)\boldsymbol{\pi}(a|s,x)dads, \quad \tilde{\sigma}^2(s,x,\boldsymbol{\pi}_{s,x})ds = \int_{\mathcal{A}}\sigma^2(s,x,a)\boldsymbol{\pi}(a|s,x)dads,$$

$$\sum_{k=1}^{\ell}\int_{[0,1]^n}\gamma_k(s,x,G(s,x,u),z)du\cdot\nu_k(dz)ds = \sum_{k=1}^{\ell}\int_{\mathcal{A}}\gamma_k\left(s,x,a,z\right)\nu_k(dz)ds\cdot\boldsymbol{\pi}(a|s,x)da.$$

Thus, the semimartingale characteristics of the exploratory state process are the averages of those of the sample state process over action randomization.

Remark 1. *In general, there may be other ways to formulate the exploratory SDE in the jump-diffusion case as we may be able to obtain alternative representations for the infinitesimal generator $\mathcal{L}^{\boldsymbol{\pi}}$ based on (15). However, the law of the exploratory state would not change because its generator stays the same.*

A technical yet foundational question is the well-posedness (i.e. existence and uniqueness of solution) of the exploratory SDE (17), which we address below. For that we first specify the class of admissible strategies, which is the same as those considered in Jia and Zhou (2023) for pure diffusions.

Definition 1. *A policy $\boldsymbol{\pi} = \boldsymbol{\pi}(\cdot|\cdot, \cdot)$ is called admissible, if*

(i) $\boldsymbol{\pi}(\cdot|t, x) \in \mathcal{P}(\mathcal{A})$, *supp*$\boldsymbol{\pi}(\cdot|t, x) = \mathcal{A}$ *for every* $(t, x) \in [0, T] \times \mathbb{R}^d$ *and* $\boldsymbol{\pi}(a|t, x) : (t, x, a) \in [0, T] \times \mathbb{R}^d \times \mathcal{A} \to \mathbb{R}$ *is measurable;*

(ii) $\boldsymbol{\pi}(a|t, x)$ *is continuous in* (t, x), *i.e.,* $\int_{\mathcal{A}} |\boldsymbol{\pi}(a|t, x) - \boldsymbol{\pi}(a|t', x')| \, da \to 0$ *as* $(t', x') \to (t, x)$. *Furthermore, for any* $K > 0$, *there is a constant* $C_K > 0$ *independent of* (t, a) *such that*

$$\int_{\mathcal{A}} \left|\boldsymbol{\pi}(a|t, x) - \boldsymbol{\pi}(a|t, x')\right| da \le C_K |x - x'|, \ \forall x, x' \in \mathbb{R}^d_K;$$

(iii) $\forall (t, x)$, $\int_{\mathcal{A}} |\log \boldsymbol{\pi}(a|t, x)| \boldsymbol{\pi}(a|t, x) da \le C(1 + |x|^p)$ *for some* $p \ge 2$ *and* C *is a positive constant; for any* $q \ge 1$, $\int_{\mathcal{A}} |a|^q \boldsymbol{\pi}(a|t, x) da \le C_q(1 + |x|^p)$ *for some* $p \ge 2$ *and* C_q *is a positive constant that can depend on* q.

Next, we establish the well-posedness of (17) under any admissible policy. The result of the next lemma regarding $\tilde{b}$ and $\tilde{\sigma}$ is provided in the proof of Lemma 2 in Jia and Zhou (2022b), which uses property (ii) of admissibility.

Lemma 1. *Under Assumption 1, for any admissible policy* $\boldsymbol{\pi}$, *the functions* $\tilde{b}(t, x, \boldsymbol{\pi}_{t,x})$ *and* $\tilde{\sigma}(t, x, \boldsymbol{\pi}_{t,x})$ *have the following properties:*

(i) (local Lipschitz continuity) for $K > 0$, *there exists a constant* $C_K > 0$ *such that* $\forall t \in [0, T], (x, x') \in \mathbb{R}^d_K$,

$$|\tilde{b}(t, x, \boldsymbol{\pi}_{t,x}) - \tilde{b}(t, x', \boldsymbol{\pi}_{t,x'})|^2 + |\tilde{\sigma}(t, x, \boldsymbol{\pi}_{t,x}) - \tilde{\sigma}(t, x', \boldsymbol{\pi}_{t,x'})|^2 \le C_K |x - x'|^2 \ ;$$

(ii) (linear growth in x*) there exists a constant* $C > 0$ *such that* $\forall (t, x) \in [0, T] \times \mathbb{R}^d$,

$$|\tilde{b}(t, x, \boldsymbol{\pi}_{t,x})|^2 + |\tilde{\sigma}(t, x, \boldsymbol{\pi}_{t,x})|^2 \le C(1 + |x|^2).$$

We now establish similar properties for $\gamma(t, x, G^{\boldsymbol{\pi}_{t,x}}(u), z)$ in the following lemmas, whose proofs are relegated to the appendix.

Lemma 2 (linear growth in x)**.** *Under Assumption 1, for any admissible* $\boldsymbol{\pi}$ *and any* $p \ge 2$, *there exists a constant* $C_p > 0$ *that can depend on* p *such that* $\forall (t, x) \in [0, T] \times \mathbb{R}^d$,

$$\sum_{k=1}^{\ell} \int_{\mathbb{R} \times [0,1]^n} |\gamma_k \left(t, x, G^{\boldsymbol{\pi}_{t,x}}(u), z\right)|^p \, \nu_k(dz) du \le C_p(1 + |x|^p).$$

For the local Lipschitz continuity of $\gamma_k(t,x,G^{\boldsymbol{\pi}_{t,x}}(u),z)$, we make an additional assumption.

Assumption 2. *For $k=1,\cdots,\ell$, the following conditions hold.*

(i) For any $K>0$ and any $p\geq 2$, there exists a constant $C_{K,p}>0$ that can depend on K and p such that

$$\int_{\mathbb{R}}\left|\gamma_k(t,x,a,z)-\gamma_k(t,x,a',z)\right|^p\nu_k(dz)\leq C_{K,p}|a-a'|^p,\quad \forall t\in[0,T],a,a'\in\mathcal{A},x\in\mathbb{R}^d_K,z\in\mathbb{R}.$$

(ii) For any $K>0$ and any $p\geq 2$, there exists a constant $C_{K,p}>0$ that can depend on K and p such that

$$\int_{[0,1]^n}\left|G^{\boldsymbol{\pi}}(t,x,u)-G^{\boldsymbol{\pi}}(t,x',u)\right|^p du\leq C_{K,p}|x-x'|^p.$$

For a stochastic feedback policy $\boldsymbol{\pi}_{t,x}\sim\mathcal{N}(\mu(t,x),A(t,x)A(t,x)^\top)$, we have $G^{\boldsymbol{\pi}}(t,x,u)=\mu(t,x)+A(t,x)\Phi^{-1}(u)$. Clearly, Assumption 2-(ii) holds provided that $\mu(t,x)$ and $A(t,x)$ are locally Lipschitz continuous in x.

Lemma 3 (local Lipschitz continuity)**.** *Under Assumptions 1 and 2, for any admissible policy $\boldsymbol{\pi}$, any $K>0$, and any $p\geq 2$, there exists a constant $C_{K,p}>0$ that can depend on K and p such that $\forall t\in[0,T],(x,x')\in\mathbb{R}^d_K$,*

$$\sum_{k=1}^{\ell}\int_{\mathbb{R}\times[0,1]^n}\left|\gamma_k\left(t,x,G^{\boldsymbol{\pi}_{t,x}}(u),z\right)-\gamma_k\left(t,x',G^{\boldsymbol{\pi}_{t,x'}}(u),z\right)\right|^p\nu_k(dz)du\leq C_{K,p}|x-x'|^p.$$

With Lemmas 1 to 3, we can now apply (Kunita 2004, Theorem 3.2) and (Situ 2006, Theorem 119) to obtain the well-posedness of (17) along with the moment estimate of its solution.

Proposition 1. *Under Assumptions 1 and 2, for any admissible policy $\boldsymbol{\pi}$, there exists a unique strong solution $\{\tilde{X}^{\boldsymbol{\pi}}_t,0\leq t\leq T\}$ to the exploratory Lévy SDE (17). Furthermore, for any $p\geq 2$, there exists a constant $C_p>0$ such that*

$$\mathbb{E}^{\bar{\mathbb{P}}}_{t,x}\left[\sup_{t\leq s\leq T}|\tilde{X}^{\boldsymbol{\pi}}_s|^p\right]\leq C_p(1+|x|^p). \tag{18}$$

It should be noted that the conditions imposed in Assumptions 1 and 2 are sufficient but not necessary for obtaining the well-posedness and moment estimate of the exploratory Lévy SDE (17). For a specific problem, weaker conditions may suffice for these results if we exploit special structures of the problem.

From the previous discussion, we see that for a given admissible stochastic feedback policy $\boldsymbol{\pi}$, the sample state process $\{X^{\boldsymbol{\pi}}_t,t\in[0,T]\}$ and the exploratory state process $\{\tilde{X}^{\boldsymbol{\pi}}_t,t\in[0,T]\}$ associated with $\boldsymbol{\pi}$ share the same infinitesimal generator and hence the same probability law. This is justified by (Ethier and Kurtz 1986, Chapter 4, Theorem 4.1) on the condition that the function space $C^{1,2}_0([0,T)\times\mathbb{R}^d)$ is a core of the generator, which we assume to hold. It follows that

$$\mathbb{E}^{\bar{\mathbb{P}}}_{t,x}\left[\sup_{t\leq s\leq T}|X^{\boldsymbol{\pi}}_s|^p\right]=\mathbb{E}^{\mathbb{P}}_{t,x}\left[\sup_{t\leq s\leq T}|\tilde{X}^{\boldsymbol{\pi}}_s|^p\right]\leq C_p(1+|x|^p) \tag{19}$$

if (18) holds.

2.3 Exploratory HJB equation

With the exploratory dynamics (17), for any admissible stochastic policy $\boldsymbol{\pi}$ the value function $J(t,x;\boldsymbol{\pi})$ given by (9) can be rewritten as

$$J(t,x;\boldsymbol{\pi}) = \mathbb{E}^{\mathbb{P}}_{t,x}\left[\int_t^T e^{-\beta(s-t)}\int_{\mathcal{A}}\left(r(s,\tilde{X}^{\boldsymbol{\pi}}_s,a^{\boldsymbol{\pi}}_s) - \theta\log\boldsymbol{\pi}(a^{\boldsymbol{\pi}}_s|s,\tilde{X}^{\boldsymbol{\pi}}_{s-})\right)\boldsymbol{\pi}(a^{\boldsymbol{\pi}}_s|s,\tilde{X}^{\boldsymbol{\pi}}_{s-})dads \right. \\ \left. + e^{-\beta(T-t)}h(\tilde{X}^{\boldsymbol{\pi}}_T)\right]. \tag{20}$$

Under Assumption 1 and using the admissibility of $\boldsymbol{\pi}$ and (18), it is easy to see that J has polynomial growth in x. We provide the Feynman–Kac formula for this function in Lemma 4 by working the representation (20). In the proof, we consider the finite and infinite jump activity cases separately because special care is needed in the latter. We revise Assumption 1 by adding one more condition for this case.

Assumption 1′. *Conditions (i) to (iv) in Assumption 1 hold. We further assume condition (v): if $\int_{\mathbb{R}}\nu_k(dz) = \infty$, $|\gamma_k(t,x,a,z)|$ is bounded for any $|z|\le 1$, $t\in[0,T]$, $x\in\mathbb{R}^d_K$, and $a\in\mathcal{A}$.*

For Lemma 4, Lemma 5 and Theorem 1, we impose Assumption 1′ and assume that the exploratory SDE (17) is well-posed with the moment estimate (18). For simplicity, we do not explicitly mention these assumptions in the statement of the results.

Lemma 4. *Given an admissible stochastic policy $\boldsymbol{\pi}$, suppose there exists a solution $\phi\in C^{1,2}([0,T)\times\mathbb{R}^d)\cap C([0,T]\times\mathbb{R}^d)$ to the following partial integro-differential equation (PIDE):*

$$\frac{\partial\phi}{\partial t}(t,x) + \int_{\mathcal{A}}\left[H(t,x,a,\phi_x,\phi_{xx},\phi) - \theta\log\boldsymbol{\pi}(a|t,x)\right]\boldsymbol{\pi}(a|t,x)da - \beta\phi(t,x) = 0,\ (t,x)\in[0,T)\times\mathbb{R}^d, \tag{21}$$

with terminal condition $\phi(T,x) = h(x)$, $x\in\mathbb{R}^d$. Moreover, for some $p\ge 2$, ϕ satisfies

$$|\phi(t,x)| \le C(1+|x|^p),\ \forall(t,x)\in[0,T]\times\mathbb{R}^d. \tag{22}$$

Then ϕ is the value function of the policy $\boldsymbol{\pi}$, i.e. $J(t,x;\boldsymbol{\pi}) = \phi(t,x)$.

For ease of presentation, we henceforth assume the value function $J(\cdot,\cdot;\boldsymbol{\pi})\in C^{1,2}([0,T)\times\mathbb{R}^d)\cap C([0,T]\times\mathbb{R}^d)$ for any admissible stochastic policy $\boldsymbol{\pi}$.

Remark 2. *The conclusion in Lemma 4 still holds if we assume $\phi_x(t,x)$ shows polynomial growth in x instead of Assumption 1′-(v).*

Next, we consider the *optimal* value function defined by

$$J^*(t,x) = \sup_{\boldsymbol{\pi}\in\boldsymbol{\Pi}} J(t,x;\boldsymbol{\pi}),$$

where $\boldsymbol{\Pi}$ is the class of admissible strategies. The following result characterizes J^* and the optimal stochastic policy through the so-called *exploratory HJB equation*.

Lemma 5. *Suppose there exists a solution $\psi \in C^{1,2}([0,T) \times \mathbb{R}^d) \cap C([0,T] \times \mathbb{R}^d)$ to the exploratory HJB equation:*

$$\frac{\partial \psi}{\partial t}(t,x) + \sup_{\boldsymbol{\pi} \in \mathcal{P}(\mathcal{A})} \int_{\mathcal{A}} \{H(t,x,a,\psi_x,\psi_{xx},\psi) - \theta \log \boldsymbol{\pi}(a|t,x)\} \boldsymbol{\pi}(a|t,x) da - \beta \psi(t,x) = 0, \quad (23)$$

with the terminal condition $\psi(T,x) = h(x)$, where H is the Hamiltonian defined in (7). Moreover, for some $p \geq 2$, ψ satisfies

$$|\psi(t,x)| \leq C(1 + |x|^p), \; \forall (t,x) \in [0,T] \times \mathbb{R}^d,$$

and it holds that

$$\int_{\mathcal{A}} \exp\left(\frac{1}{\theta} H(t,x,a,\psi_x,\psi_{xx},\psi)\right) da < \infty.$$

Then, the Gibbs measure or Boltzman distribution

$$\boldsymbol{\pi}^*(a|t,x) \propto \exp\left(\frac{1}{\theta} H(t,x,a,\psi_x,\psi_{xx},\psi)\right) \quad (24)$$

is the optimal stochastic policy and $J^*(t,x) = \psi(t,x)$ provided that $\boldsymbol{\pi}^*$ is admissible.

Plugging the optimal stochastic policy (24) in Equation (23) to remove the supremum operator, we obtain the following nonlinear PIDE for the optimal value function J^*:

$$\frac{\partial J^*}{\partial t}(t,x) + \theta \log \left[\int_{\mathcal{A}} \exp\left(\frac{1}{\theta} H(t,x,a,J^*_x,J^*_{xx},J^*)\right) da \right] - \beta J^*(t,x) = 0; \quad J^*(T,x) = h(x).$$

3 q-Learning Theory

3.1 q-function and policy improvement

We present the q-learning theory for jump-diffusions that includes both policy evaluation and policy improvement, now that the exploratory formulation has been set up. The theory can be developed similarly to Jia and Zhou (2023); so we will just highlight the main differences in the analysis, skipping the parts that are similar.

Definition 2. *The q-function of the problem (1)–(2) associated with a given policy $\boldsymbol{\pi} \in \boldsymbol{\Pi}$ is defined by*

$$q(t,x,a;\boldsymbol{\pi}) = \frac{\partial J(t,x;\boldsymbol{\pi})}{\partial t} + H(t,x,a,J_x,J_{xx},J) - \beta J(t,x;\boldsymbol{\pi}), \;\; (t,x,a) \in [0,T] \times \mathbb{R}^d \times \mathcal{A},$$

where J is given in (20) and the Hamiltonian function H is defined in (7).

It is an immediate consequence of Lemma 4 that the q-function satisfies

$$\int_{\mathcal{A}} [q(t,x,a;\boldsymbol{\pi}) - \theta \log \boldsymbol{\pi}(a|t,x)] \boldsymbol{\pi}(a|t,x) da = 0, \;\; (t,x) \in [0,T] \times \mathbb{R}^d.$$

The following policy improvement theorem can be proved similarly as in (Jia and Zhou 2023, Theorem 2) by using the arguments in the proof of Lemma 4.

Theorem 1 (Policy Improvement)**.** *For any given $\boldsymbol{\pi} \in \boldsymbol{\Pi}$, define*

$$\boldsymbol{\pi}'(\cdot|t,x) \propto \exp\left(\frac{1}{\gamma}H(t,x,\cdot,J_x(t,x;\boldsymbol{\pi}),J_{xx}(t,x;\boldsymbol{\pi}),J(t,\cdot;\boldsymbol{\pi}))\right) \propto \exp\left(\frac{1}{\gamma}q(t,x,\cdot;\boldsymbol{\pi})\right).$$

If $\boldsymbol{\pi}' \in \boldsymbol{\Pi}$, then

$$J(t,x,\boldsymbol{\pi}') \geq J(t,x,\boldsymbol{\pi}).$$

Moreover, if the following map

$$\begin{aligned}\mathcal{I}(\boldsymbol{\pi}) &= \frac{\exp\left(\frac{1}{\theta}H(t,x,\cdot,J_x(t,x;\boldsymbol{\pi}),J_{xx}(t,x;\boldsymbol{\pi}),J(t,\cdot;\boldsymbol{\pi}))\right)}{\int_{\mathcal{A}}\exp\left(\frac{1}{\theta}H(t,x,a,J_x(t,x;\boldsymbol{\pi}),J_{xx}(t,x;\boldsymbol{\pi}),J(t,\cdot;\boldsymbol{\pi}))\right)da}, \quad \boldsymbol{\pi} \in \boldsymbol{\Pi} \\ &= \frac{\exp\left(\frac{1}{\theta}q(t,x,\cdot;\boldsymbol{\pi})\right)}{\int_{\mathcal{A}}\exp\left(\frac{1}{\theta}q(t,x,a;\boldsymbol{\pi})\right)da}\end{aligned}$$

has a fixed point $\boldsymbol{\pi}^*$, then $\boldsymbol{\pi}^*$ is an optimal policy.

3.2 Martingale characterization of the q-function

Next we derive the martingale characterization of the q-function associated with a policy $\boldsymbol{\pi} \in \boldsymbol{\Pi}$, assuming that its value function has already been learned and known. We will highlight the major differences in the proof, provided in the appendix, compared with the pure diffusion setting. For Theorem 2, Theorem 3, and Theorem 4, we impose Assumption 1 and assume the moment estimate (19) for the sample state process holds without explicitly mentioning them in the theorem statements.

Theorem 2. *Let a policy $\boldsymbol{\pi} \in \boldsymbol{\Pi}$, its value function $J(\cdot,\cdot;\boldsymbol{\pi}) \in C^{1,2}([0,T) \times \mathbb{R}^d) \cap C([0,T] \times \mathbb{R}^d)$ and a continuous function $\hat{q} : [0,T] \times \mathbb{R}^d \times \mathcal{A} \to \mathbb{R}$ be given. Assume that $J(t,x;\boldsymbol{\pi})$ and $J_x(t,x;\boldsymbol{\pi})$ both have polynomial growth in x. Then the following results hold.*

(i) $\hat{q}(t,x,a) = q(t,x,a;\boldsymbol{\pi})$ for all (t,x,a) if and only if for any (t,x), the following process

$$e^{-\beta s}J(t,X_s^{\boldsymbol{\pi}};\boldsymbol{\pi}) + \int_t^s e^{-\beta\tau}[r(\tau,X_\tau^{\boldsymbol{\pi}},a_\tau^{\boldsymbol{\pi}}) - \hat{q}(\tau,X_\tau^{\boldsymbol{\pi}},a_\tau^{\boldsymbol{\pi}})]d\tau \tag{25}$$

is a $(\{\mathcal{G}_s\}_{s\geq 0}, \bar{\mathbb{P}})$-martingale, where $X^{\boldsymbol{\pi}} = \{X_s^{\boldsymbol{\pi}} : t \leq s \leq T\}$ is the sample state process defined in (8) with $X_t^{\boldsymbol{\pi}} = x$.

(ii) If $\hat{q}(t,x,a) = q(t,x,a;\boldsymbol{\pi})$ for all (t,x,a), then for any $\boldsymbol{\pi}' \in \boldsymbol{\Pi}$ and any (t,x), the following process

$$e^{-\beta s}J(t,X_s^{\boldsymbol{\pi}'};\boldsymbol{\pi}) + \int_t^s e^{-\beta\tau}[r(\tau,X_\tau^{\boldsymbol{\pi}'},a_\tau^{\boldsymbol{\pi}'}) - \hat{q}(\tau,X_\tau^{\boldsymbol{\pi}'},a_\tau^{\boldsymbol{\pi}'})]d\tau \tag{26}$$

is a $(\{\mathcal{G}_s\}_{s\geq 0}, \bar{\mathbb{P}})$-martingale, where $\{X_s^{\boldsymbol{\pi}'} : t \leq s \leq T\}$ is the solution to (8) under $\boldsymbol{\pi}'$ with initial condition $X_t^{\boldsymbol{\pi}'} = x$.

(iii) If there exists $\boldsymbol{\pi}' \in \boldsymbol{\Pi}$ such that for all (t,x), the process (26) is a $(\{\mathcal{G}_s\}_{s\geq 0}, \bar{\mathbb{P}})$-martingale where $X_t^{\boldsymbol{\pi}'} = x$, then we have $\hat{q}(t,x,a) = q(t,x,a;\boldsymbol{\pi})$ for all (t,x,a).

Moreover, in any of the three cases above, the q-function satisfies

$$\int_{\mathcal{A}} \{q(t,x,a;\boldsymbol{\pi}) - \gamma \log \boldsymbol{\pi}(a|t,x)\}\boldsymbol{\pi}(a|t,x)da = 0, \quad \textit{for all } (t,x) \in [0,T] \times \mathbb{R}^d.$$

Remark 3. *Similar to Jia and Zhou (2023), Theorem 2-(i) facilitates on-policy learning, where learning the q-function of the given target policy $\boldsymbol{\pi}$ is based on data $\{(s, X_s^{\boldsymbol{\pi}}, a_s^{\boldsymbol{\pi}}), t \leq s \leq T\}$ generated by $\boldsymbol{\pi}$. On the other hand, Theorem 2-(ii) and -(iii) are for off-policy learning, where learning the q-function of $\boldsymbol{\pi}$ is based on data generated by a different, called behavior, policy $\boldsymbol{\pi}'$.*

Next, we extend Theorem 7 in Jia and Zhou (2023) and obtain a martingale characterization of the value function and the q-function simultaneously. The proof is essentially the same and hence omitted.

Theorem 3. *Let a policy $\boldsymbol{\pi} \in \boldsymbol{\Pi}$, a function $\hat{J} \in C^{1,2}([0,T) \times \mathbb{R}^d) \cap C([0,T] \times \mathbb{R}^d)$ with polynomial growth and a continuous function $\hat{q} : [0,T] \times \mathbb{R}^d \times \mathcal{A} \to \mathbb{R}$ be given satisfying*

$$\hat{J}(T,x) = h(x), \quad \int_{\mathcal{A}} \{\hat{q}(t,x,a) - \theta \log \boldsymbol{\pi}(a|t,x)\}\boldsymbol{\pi}(a|t,x)da = 0, \quad \textit{for all } (t,x) \in [0,T] \times \mathbb{R}^d.$$

Assume that $\hat{J}$ and $\hat{J}_x$ both have polynomial growth. Then

(i) $\hat{J}$ and $\hat{q}$ are respectively the value function and the q-function associated with $\boldsymbol{\pi}$ if and only if for all $(t,x) \in [0,T] \times \mathbb{R}^d$, the following process

$$e^{-\beta s}\hat{J}(s, X_s^{\boldsymbol{\pi}}) + \int_t^s e^{-\beta\tau}[r(\tau, X_\tau^{\boldsymbol{\pi}}, a_\tau^{\boldsymbol{\pi}}) - \hat{q}(\tau, X_\tau^{\boldsymbol{\pi}}, a_\tau^{\boldsymbol{\pi}})]d\tau$$

is a $(\{\mathcal{G}_s\}_{s\geq 0}, \bar{\mathbb{P}})$-martingale, where $X^{\boldsymbol{\pi}} = \{X_s^{\boldsymbol{\pi}} : t \leq s \leq T\}$ satisfies (8) with $X_t^{\boldsymbol{\pi}} = x$.

(ii) If $\hat{J}$ and $\hat{q}$ are respectively the value function and the q-function associated with $\boldsymbol{\pi}$, then for any $\boldsymbol{\pi}' \in \boldsymbol{\Pi}$ and for all $(t,x) \in [0,T] \times \mathbb{R}^d$, the following process

$$e^{-\beta s}\hat{J}(s, X_s^{\boldsymbol{\pi}'}) + \int_t^s e^{-\beta\tau}[r(\tau, X_\tau^{\boldsymbol{\pi}'}, a_\tau^{\boldsymbol{\pi}'}) - \hat{q}(\tau, X_\tau^{\boldsymbol{\pi}'}, a_\tau^{\boldsymbol{\pi}'})]d\tau \tag{27}$$

is a $(\{\mathcal{G}_s\}_{s\geq 0}, \bar{\mathbb{P}})$-martingale, where $\{X_s^{\boldsymbol{\pi}'} : t \leq s \leq T\}$ satisfies (8) with $X_t^{\boldsymbol{\pi}'} = x$.

(iii) If there exists $\boldsymbol{\pi}' \in \boldsymbol{\Pi}$ such that for all (t,x), the process (27) is a $(\{\mathcal{G}_s\}_{s\geq 0}, \bar{\mathbb{P}})$-martingale where $X_t^{\boldsymbol{\pi}'} = x$, then we have $\hat{J}(t,x) = J(t,x;\boldsymbol{\pi})$ and $\hat{q}(t,x,a) = q(t,x,a;\boldsymbol{\pi})$ for all (t,x,a).

In any of the three cases above, if it holds that $\boldsymbol{\pi}(a|t,x) = \frac{\exp(\frac{1}{\theta}\hat{q}(t,x,a))}{\int_{\mathcal{A}} \exp(\frac{1}{\theta}\hat{q}(t,x,a))da}$, then $\boldsymbol{\pi}$ is the optimal policy and $\hat{J}$ is the optimal value function.

3.3 Optimal q-function

We consider in this section the optimal q-function, i.e., the q-function associated with the optimal policy $\boldsymbol{\pi}^*$ in (24). Based on Definition 2, we can define it by

$$q^*(t,x,a) = \frac{\partial J^*(t,x)}{\partial t} + H(t,x,a,J_x^*,J_{xx}^*,J^*) - \beta J^*(t,x),$$

where J^* is the optimal value function that solves the exploratory HJB equation in (23).

The following is the martingale condition that characterize the optimal value function J^* and the optimal q-function, that can be proved analogously to Theorem 9 in Jia and Zhou (2023).

Theorem 4. *Let a function $\hat{J}^* \in C^{1,2}([0,T) \times \mathbb{R}^d) \cap C([0,T] \times \mathbb{R}^d)$ and a continuous function $\hat{q}^* : [0,T] \times \mathbb{R}^d \times \mathcal{A} \to \mathbb{R}$ be given satisfying*

$$\hat{J}^*(T,x) = h(x), \quad \int_{\mathcal{A}} \exp\left(\frac{1}{\theta}\hat{q}^*(t,x,a)\right) da = 1, \quad \textit{for all } (t,x) \in [0,T] \times \mathbb{R}^d.$$

Assume that $\hat{J}^*(t,x)$ and $\hat{J}^*_x(t,x)$ both have polynomial growth in x. Then

(i) If $\hat{J}^*$ and $\hat{q}^*$ are respectively the optimal value function and the optimal q-function, then for any $\boldsymbol{\pi} \in \boldsymbol{\Pi}$ and for all $(t,x) \in [0,T] \times \mathbb{R}^d$, the following process

$$e^{-\beta s}\hat{J}^*(s, X_s^{\boldsymbol{\pi}}) + \int_t^s e^{-\beta\tau}[r(\tau, X_\tau^{\boldsymbol{\pi}}, a_\tau^{\boldsymbol{\pi}}) - \hat{q}^*(\tau, X_\tau^{\boldsymbol{\pi}}, a_\tau^{\boldsymbol{\pi}})]d\tau \tag{28}$$

is a $(\{\mathcal{G}_s\}_{s\geq 0}, \bar{\mathbb{P}})$-martingale, where $X^{\boldsymbol{\pi}} = \{X_s^{\boldsymbol{\pi}} : t \leq s \leq T\}$ satisfies (8) with $X_t^{\boldsymbol{\pi}} = x$. Moreover, in this case, $\hat{\boldsymbol{\pi}}^*(a|t,x) = \exp\left(\frac{1}{\theta}\hat{q}^*(t,x,a)\right)$ is the optimal stochastic policy.

(ii) If there exists $\boldsymbol{\pi} \in \boldsymbol{\Pi}$ such that for all (t,x), the process (28) is a $(\{\mathcal{G}_s\}_{s\geq 0}, \bar{\mathbb{P}})$-martingale where $X_t^{\boldsymbol{\pi}} = x$, then $\hat{J}^*$ and $\hat{q}^*$ are respectively the optimal value function and the optimal q-function.

4 q-Learning Algorithms

In this section we present learning algorithms based on the martingale characterization of the q-function discussed in the previous section. We need to distinguish two cases, depending on whether or not the density function of the Gibbs measure generated from the q-function can be computed and integrated explicitly.

We first discuss the case when the normalizing constant in the Gibbs measure can be computed explicitly. We denote by J^ψ and q^ϕ the parameterized function approximators for the optimal value function and optimal q-function, respectively. In view of Theorem 4, these approximators are chosen to satisfy

$$J^\psi(T,x) = h(x), \ \int_{\mathcal{A}} \exp\left(\frac{1}{\theta}q^\phi(t,x,a)\right) da = 1. \tag{29}$$

We can then update (ψ, ϕ) by enforcing the martingale condition discussed in Theorem 4 and applying the techniques developed in Jia and Zhou (2022a). This procedure has been discussed in details in Section 4.1 of Jia and Zhou (2023), and hence we omit the details. For reader's convenience, we present Algorithms 1 and 2, which summarize the offline and online q-learning algorithms respectively. Such algorithms are based on the so-called martingale orthogonality condition in Jia and Zhou (2022a), where the typical choices of test functions in these algorithms are $\xi_t = \frac{\partial J^\psi}{\partial \psi}(t, X_t^{\boldsymbol{\pi}^\phi})$, and $\zeta_t = \frac{\partial q^\phi}{\partial \phi}(t, X_t^{\boldsymbol{\pi}^\phi}, a_t^{\boldsymbol{\pi}^\phi})$, where $\boldsymbol{\pi}^\phi$ is the policy generated by q^ϕ. Note that these two algorithms are *identical* to Algorithms 2 and 3 in Jia and Zhou (2023).

When the normalizing constant in the Gibbs measure is not available, we take the same approach as in Jia and Zhou (2023) to develop learning algorithms. Specifically, we consider $\{\boldsymbol{\pi}^\phi(\cdot|t,x)\}_{\phi\in\Phi}$,

Algorithm 1 Offline–Episodic q-Learning Algorithm

Inputs: initial state x_0, horizon T, time step Δt, number of episodes N, number of mesh grids K, initial learning rates α_ψ, α_ϕ and a learning rate schedule function $l(\cdot)$ (a function of the number of episodes), functional forms of parameterized value function $J^\psi(\cdot,\cdot)$ and q-function $q^\phi(\cdot,\cdot,\cdot)$ satisfying (29), functional forms of test functions $\boldsymbol{\xi}(t, x_{\cdot\wedge t}, a_{\cdot\wedge t})$ and $\boldsymbol{\zeta}(t, x_{\cdot\wedge t}, a_{\cdot\wedge t})$, and temperature parameter θ.

Required program (on-policy): environment simulator $(x', r) = \mathit{Environment}_{\Delta t}(t, x, a)$ that takes current time–state pair (t, x) and action a as inputs and generates state x' at time $t + \Delta t$ and instantaneous reward r at time t as outputs. Policy $\boldsymbol{\pi}^\phi(a|t, x) = \exp\left(\frac{1}{\gamma} q^\phi(t, x, a)\right)$.

Required program (off-policy): observations $\{a_{t_k}, r_{t_k}, x_{t_{k+1}}\}_{k=0,\cdots,K-1} \cup \{x_{t_K}, h(x_{t_K})\} = \mathit{Obervation}(\Delta t)$ including the observed actions, rewards, and state trajectories under the given behavior policy at the sampling time grid with step size Δt.

Learning procedure:

Initialize ψ, ϕ.

for episode $j = 1$ **to** N **do**

Initialize $k = 0$. Observe initial state x_0 and store $x_{t_k} \leftarrow x_0$.

{**On-policy case**

while $k < K$ **do**

Generate action $a_{t_k} \sim \boldsymbol{\pi}^\phi(\cdot|t_k, x_{t_k})$.

Apply a_{t_k} to environment simulator $(x, r) = \mathit{Environment}_{\Delta t}(t_k, x_{t_k}, a_{t_k})$, and observe new state x and reward r as outputs. Store $x_{t_{k+1}} \leftarrow x$ and $r_{t_k} \leftarrow r$.

Update $k \leftarrow k + 1$.

end while

}

{**Off-policy case** Obtain one observation $\{a_{t_k}, r_{t_k}, x_{t_{k+1}}\}_{k=0,\cdots,K-1} \cup \{x_{t_K}, h(x_{t_K})\} = \mathit{Obervation}(\Delta t)$.

}

For every $k = 0, 1, \cdots, K-1$, compute and store test functions $\xi_{t_k} = \boldsymbol{\xi}(t_k, x_{t_0}, \cdots, x_{t_k}, a_{t_0}, \cdots, a_{t_k})$, $\zeta_{t_k} = \boldsymbol{\zeta}(t_k, x_{t_0}, \cdots, x_{t_k}, a_{t_0}, \cdots, a_{t_k})$.

Compute

$$\Delta\psi = \sum_{i=0}^{K-1} \xi_{t_i} \big[J^\psi(t_{i+1}, x_{t_{i+1}}) - J^\psi(t_i, x_{t_i}) + r_{t_i}\Delta t - q^\phi(t_i, x_{t_i}, a_{t_i})\Delta t - \beta J^\psi(t_i, x_{t_i})\Delta t\big],$$

$$\Delta\phi = \sum_{i=0}^{K-1} \zeta_{t_i} \big[J^\psi(t_{i+1}, x_{t_{i+1}}) - J^\psi(t_i, x_{t_i}) + r_{t_i}\Delta t - q^\phi(t_i, x_{t_i}, a_{t_i})\Delta t - \beta J^\psi(t_i, x_{t_i})\Delta t\big].$$

Update ψ and ϕ by

$$\psi \leftarrow \psi + l(j)\alpha_\psi \Delta\psi.$$

$$\phi \leftarrow \phi + l(j)\alpha_\phi \Delta\phi.$$

end for

Algorithm 2 Online-Incremental q-Learning Algorithm

Inputs: initial state x_0, horizon T, time step Δt, number of mesh grids K, initial learning rates α_ψ, α_ϕ and learning rate schedule function $l(\cdot)$ (a function of the number of episodes), functional forms of parameterized value function $J^\psi(\cdot,\cdot)$ and q-function $q^\phi(\cdot,\cdot,\cdot)$ satisfying (29), functional forms of test functions $\boldsymbol{\xi}(t, x_{\cdot\wedge t}, a_{\cdot\wedge t})$ and $\boldsymbol{\zeta}(t, x_{\cdot\wedge t}, a_{\cdot\wedge t})$, and temperature parameter θ.

Required program (on-policy): environment simulator $(x', r) = \mathit{Environment}_{\Delta t}(t, x, a)$ that takes current time–state pair (t, x) and action a as inputs and generates state x' at time $t + \Delta t$ and instantaneous reward r at time t as outputs. Policy $\boldsymbol{\pi}^\phi(a|t, x) = \exp\left(\frac{1}{\theta} q^\phi(t, x, a)\right)$.

Required program (off-policy): observations $\{a, r, x'\} = \mathit{Obervation}(t, x; \Delta t)$ including the observed actions, rewards, and state when the current time-state pair is (t, x) under the given behavior policy at the sampling time grid with step size Δt.

Learning procedure:

Initialize ψ, ϕ.
for episode $j = 1$ **to** ∞ **do**
  Initialize $k = 0$. Observe initial state x_0 and store $x_{t_k} \leftarrow x_0$.
  while $k < K$ **do**
    {**On-policy case**
    Generate action $a_{t_k} \sim \boldsymbol{\pi}^\phi(\cdot|t_k, x_{t_k})$.
    Apply a_{t_k} to environment simulator $(x, r) = \mathit{Environment}_{\Delta t}(t_k, x_{t_k}, a_{t_k})$, and observe new state x and reward r as outputs. Store $x_{t_{k+1}} \leftarrow x$ and $r_{t_k} \leftarrow r$.
    }
    {**Off-policy case**
    Obtain one observation $a_{t_k}, r_{t_k}, x_{t_{k+1}} = \mathit{Obervation}(t_k, x_{t_k}; \Delta t)$.
    }
    Compute test functions $\xi_{t_k} = \boldsymbol{\xi}(t_k, x_{t_0}, \cdots, x_{t_k}, a_{t_0}, \cdots, a_{t_k})$, $\zeta_{t_k} = \boldsymbol{\zeta}(t_k, x_{t_0}, \cdots, x_{t_k}, a_{t_0}, \cdots, a_{t_k})$.
    Compute

$$
\begin{aligned}
&\delta = J^\psi(t_{k+1}, x_{t_{k+1}}) - J^\psi(t_k, x_{t_k}) + r_{t_k}\Delta t - q^\phi(t_k, x_{t_k}, a_{t_k})\Delta t - \beta J^\psi(t_k, x_{t_k})\Delta t,\\
&\Delta\psi = \xi_{t_k}\delta,\\
&\Delta\phi = \zeta_{t_k}\delta.
\end{aligned}
$$

    Update ψ and ϕ by

$$
\psi \leftarrow \psi + l(j)\alpha_\psi \Delta\psi.
$$

$$
\phi \leftarrow \phi + l(j)\alpha_\phi \Delta\phi.
$$

    Update $k \leftarrow k + 1$
  end while
end for

which is a family of density functions of some tractable distributions, e.g. multivariate normal distributions. Starting from a stochastic policy $\boldsymbol{\pi}^{\phi}$ in this family, we update the policy by considering the optimization problem

$$\min_{\phi' \in \Phi} \mathrm{KL}\left(\boldsymbol{\pi}^{\phi'}(\cdot|t,x) \Big| \exp\left(\frac{1}{\theta} q(t,x,\cdot;\boldsymbol{\pi}^{\phi}) \right) \right).$$

Specifically, using gradient descent, we can update ϕ as in Jia and Zhou (2023), by

$$\phi \leftarrow \phi - \theta \alpha_{\phi} dt \left[\log \boldsymbol{\pi}^{\phi}(a_t^{\boldsymbol{\pi}^{\phi}}|t, X_t^{\boldsymbol{\pi}^{\phi}}) - \frac{1}{\theta} q(t, X_t^{\boldsymbol{\pi}^{\phi}}, a_t^{\boldsymbol{\pi}^{\phi}}; \boldsymbol{\pi}^{\phi}) \right] \frac{\partial}{\partial \phi} \log \boldsymbol{\pi}^{\phi}(a_t^{\boldsymbol{\pi}^{\phi}}|t, X_t^{\boldsymbol{\pi}^{\phi}}). \tag{30}$$

In the above updating rule, we need only the *values* of the $q-$function along the trajectory – the "data" – $\{(t, X_t^{\boldsymbol{\pi}^{\phi}}, a_t^{\boldsymbol{\pi}^{\phi}}); 0 \le t \le T\}$, instead of its full functional form. These values can be learned through the "temporal difference" of the value function along the data. To see this, applying Itô's formula (5) to $J(\cdot,\cdot;\boldsymbol{\pi}^{\phi})$, we have

$$\begin{aligned} q(t, X_t^{\boldsymbol{\pi}^{\phi}}, a_t^{\boldsymbol{\pi}^{\phi}}; \boldsymbol{\pi}^{\phi}) dt = {} & dJ(t, X_t^{\boldsymbol{\pi}^{\phi}}; \boldsymbol{\pi}^{\phi}) + [r(t, X_t^{\boldsymbol{\pi}^{\phi}}, a_t^{\boldsymbol{\pi}^{\phi}}) - \beta J(t, X_t^{\boldsymbol{\pi}^{\phi}}; \boldsymbol{\pi}^{\phi})] dt \\ & + J(t, X_{t-}^{\boldsymbol{\pi}^{\phi}}; \boldsymbol{\pi}^{\phi}) \sigma(t, X_{t-}^{\boldsymbol{\pi}^{\phi}}, a_t^{\boldsymbol{\pi}^{\phi}}) dW_t \\ & + \int_{\mathbb{R}^d} \left(J(t, X_{u-}^{\boldsymbol{\pi}^{\phi}} + \gamma(t, X_{u-}^{\boldsymbol{\pi}^{\phi}}, a_t^{\boldsymbol{\pi}^{\phi}}, z)) - J(t, X_{t-}^{\boldsymbol{\pi}^{\phi}}; \boldsymbol{\pi}^{\phi}) \right) \widetilde{N}(dt, dz). \end{aligned}$$

We may ignore the dW_t and $\widetilde{N}(dt,dz)$ terms which are martingale differences with mean zero, and then the updating rule in (30) becomes

$$\begin{aligned} \phi \leftarrow \phi + \alpha_{\phi} & \left[-\theta \log \boldsymbol{\pi}^{\phi}(a_t^{\boldsymbol{\pi}^{\phi}}|t, X_t^{\boldsymbol{\pi}^{\phi}}) dt + dJ(t, X_t^{\boldsymbol{\pi}^{\phi}}; \boldsymbol{\pi}^{\phi}) + \left(r(t, X_t^{\boldsymbol{\pi}^{\phi}}, a_t^{\boldsymbol{\pi}^{\phi}}) - \beta J(t, X_t^{\boldsymbol{\pi}^{\phi}}; \boldsymbol{\pi}^{\phi}) \right) dt \right] \\ & \cdot \frac{\partial}{\partial \phi} \log \boldsymbol{\pi}^{\phi}(a_t^{\boldsymbol{\pi}^{\phi}}|t, X_t^{\boldsymbol{\pi}^{\phi}}). \end{aligned}$$

Using $J^{\psi}(\cdot,\cdot)$ as the parameterized function approximator for $J(\cdot,\cdot;\boldsymbol{\pi}^{\phi})$, we arrive at the updating rule for the policy parameter ϕ :

$$\begin{aligned} \phi \leftarrow \phi + \alpha_{\phi} & \left[-\theta \log \boldsymbol{\pi}^{\phi}(a_t^{\boldsymbol{\pi}^{\phi}}|t, X_t^{\boldsymbol{\pi}^{\phi}}) dt + dJ^{\psi}(t, X_t^{\boldsymbol{\pi}^{\phi}}) + \left(r(t, X_t^{\boldsymbol{\pi}^{\phi}}, a_t^{\boldsymbol{\pi}^{\phi}}) - \beta J^{\psi}(t, X_t^{\boldsymbol{\pi}^{\phi}}) \right) dt \right] \\ & \cdot \frac{\partial}{\partial \phi} \log \boldsymbol{\pi}^{\phi}(a_t^{\boldsymbol{\pi}^{\phi}}|t, X_t^{\boldsymbol{\pi}^{\phi}}). \end{aligned}$$

Therefore, we can update ψ using the PE methods in Jia and Zhou (2022a), and update ϕ using the above rule, leading to actor–critic type of algorithms.

To conclude, we are able to use the *same* RL algorithms to learn the optimal policy and optimal value function, *without having to know a priori whether the unknown environment entails a pure diffusion or a jump-diffusion.* This important conclusion is based on the theoretical analysis carried out in the previous sections.

5 Application to Mean–Variance Portfolio Selection

We now present an applied example of the general theory and algorithms derived. Consider investing in a market where there are a risk-free asset and a risky asset (e.g., a stock or an index). The risk-free rate is $r_f > 0$ and the risky asset price process follows

$$dS_t = S_{t-}\left[\mu dt + \sigma dW_t + \int_{\mathbb{R}} (\exp(z) - 1)\widetilde{N}(dt, dz)\right].$$

Let X_t be the discounted wealth value at time t, and a_t is the discounted dollar value of the investment in the risky asset. The dynamics of a self-financing discounted wealth process is given by

$$dX_t^a = a_t\sigma\rho dt + a_t\sigma dW_t + a_t \int_{\mathbb{R}} (\exp(z) - 1)\widetilde{N}(dt, dz),$$

where $\rho := (\mu - r_f)/\sigma$. We assume $\int_{\mathbb{R}} \nu(dz) < \infty$ and

$$\int_{|z|>1} \exp(z)\nu(dz) < \infty, \quad \int_{|z|>1} \exp(2z)\nu(dz) < \infty. \tag{31}$$

Condition (31) implies that $\mathbb{E}[S_t]$ and $\mathbb{E}[S_t^2]$ are finite for every $t \geq 0$; see (Cont and Tankov 2004, Proposition 3.14). We set

$$\sigma_J^2 := \int_{\mathbb{R}} (\exp(z) - 1)^2 \nu(dz),$$

which is finite by condition (31).

Fix the investment horizon as $[0, T]$. The mean-variance (MV) portfolio selection problem considers

$$\min_a \mathrm{Var}\left[X_T^a\right] \quad \text{subject to } \mathbb{E}\left[X_T^a\right] = z.$$

We seek the optimal pre-committed strategy for the MV problem as in Zhou and Li (2000). We can transform the above constrained problem into an unconstrained one by introducing a Lagrange multiplier, which yields

$$\min_a \mathbb{E}\left[(X_T^a)^2\right] - z^2 - 2\omega\left(\mathbb{E}\left[X_T^a\right] - z\right) = \min_a \mathbb{E}\left[(X_T^a - \omega)^2\right] - (\omega - z)^2. \tag{32}$$

Note that the optimal solution to the unconstrained minimization problem depends on ω, and we can obtain the optimal multiplier ω^* by solving $\mathbb{E}\left[X_T^{a^*}(\omega)\right] = z$.

The exploratory formulation of the problem is

$$\min_{\boldsymbol{\pi}} \mathbb{E}_{t,x}^{\bar{\mathbb{P}}}\left[(X_T^{\boldsymbol{\pi}} - \omega)^2 + \theta \int_t^T \log \boldsymbol{\pi}(a_s^{\boldsymbol{\pi}} | s, X_{s-}^{\boldsymbol{\pi}}) ds\right] - (\omega - z)^2, \tag{33}$$

where the discounted wealth under a stochastic policy $\boldsymbol{\pi}$ follows

$$dX_s^{\boldsymbol{\pi}} = a_s^{\boldsymbol{\pi}}\sigma\rho ds + a_s^{\boldsymbol{\pi}}\sigma dW_s + a_s^{\boldsymbol{\pi}} \int_{\mathbb{R}} (\exp(z) - 1)\widetilde{N}(ds, dz), \quad s \in [t, T], \ X_s^{\boldsymbol{\pi}} = x.$$

5.1 Solution of the exploratory control problem

We consider the HJB equation for problem (33):

$$\frac{\partial V}{\partial t}(t,x) + \inf_{\boldsymbol{\pi}\in\mathcal{P}(\mathbb{R})} \int_{\mathbb{R}} \{H(t,x,a,V_x,V_{xx},V) + \theta \log \boldsymbol{\pi}(a|t,x)\}\boldsymbol{\pi}(a|t,x)da = 0, \tag{34}$$

with the terminal condition $V(T,x) = (x-\omega)^2 - (\omega - z)^2$. Note that supremum becomes infimum and the sign before $\theta \log \boldsymbol{\pi}(a|t,x)$ flips compared with (23) because we consider minimization here. The Hamiltonian of the problem is given by

$$\begin{aligned} H(t,x,a,V_x,V_{xx},V) &= a\sigma\rho V_x(t,x) + \frac{1}{2}a^2\sigma^2 V_{xx}(t,x) \\ &\quad + \int_{\mathbb{R}} \left(V(t,x+\gamma(a,z)) - V(t,x) - \gamma(a,z)V_x(t,x)\right)\nu(dz), \end{aligned}$$

where $\gamma(a,z) = a(e^z - 1)$. We take the following Ansatz for the solution of the HJB equation (34):

$$V(t,x) = (x-\omega)^2 f(t) + g(t) - (\omega - z)^2. \tag{35}$$

As $V(t,x)$ is quadratic in x, we can easily calculate the integral term in the Hamiltonian and obtain

$$H(t,x,a,V_x,V_{xx},V) = a\sigma\rho V_x(t,x) + \frac{1}{2}a^2(\sigma^2+\sigma_J^2)V_{xx}(t,x). \tag{36}$$

The probability density function that minimizes the integral in (34) is given by

$$\boldsymbol{\pi}_c(\cdot|t,x) \propto \exp\left(-\frac{1}{\theta}H(t,x,a,V_x,V_{xx},V)\right),$$

which is a candidate for the optimal stochastic policy. From (36), we obtain

$$\boldsymbol{\pi}_c(\cdot|t,x) \sim \mathcal{N}\left(\cdot \mid -\frac{\sigma\rho V_x}{(\sigma^2+\sigma_J^2)V_{xx}}, \frac{\theta}{(\sigma^2+\sigma_J^2)V_{xx}}\right).$$

Substituting it back to the HJB equation (34), we obtain a nonlinear PDE as

$$\begin{aligned} &V_t - \frac{\rho^2\sigma^2}{2(\sigma^2+\sigma_J^2)}\frac{(V_x)^2}{V_{xx}} - \frac{\theta}{2}\ln\frac{2\pi\theta}{(\sigma^2+\sigma_J^2)V_{xx}} = 0, \quad (t,x)\in[0,T)\times\mathbb{R}, \\ &V(T,x) = (x-\omega)^2 - (\omega-z)^2. \end{aligned}$$

We plug in the Ansatz (35) to the above PDE and obtain that $f(t)$ satisfies

$$f'(t) - \frac{\rho^2\sigma^2}{\sigma^2+\sigma_J^2}f(t) = 0,\ f(T) = 1,$$

and $g(t)$ satisfies

$$g'(t) - \frac{\theta}{2}\ln\frac{\pi\theta}{(\sigma^2+\sigma_J^2)f(t)} = 0,\ g(T) = 0.$$

These two ordinary differential equations can be solved analytically, and we obtain

$$
\begin{aligned}
V(t,x) =&(x-\omega)^2 \exp\left(-\frac{\rho^2\sigma^2}{\sigma^2+\sigma_J^2}(T-t)\right) + \frac{\theta\rho^2\sigma^2}{4(\sigma^2+\sigma_J^2)}(T^2-t^2) \\
&-\frac{\theta}{2}\left(\frac{\rho^2\sigma^2}{\sigma^2+\sigma_J^2}T + \ln\frac{\pi\theta}{\sigma^2+\sigma_J^2}\right)(T-t) - (\omega-z)^2.
\end{aligned}
$$

It follows that

$$
\boldsymbol{\pi}_c(\cdot|t,x) \sim \mathcal{N}\left(\cdot \mid -\frac{\sigma\rho}{\sigma^2+\sigma_J^2}(x-\omega), \frac{\theta}{2(\sigma^2+\sigma_J^2)}\exp\left(\frac{\rho^2\sigma^2}{\sigma^2+\sigma_J^2}(T-t)\right)\right).
$$

It is straightforward to verify that $\boldsymbol{\pi}_c$ is admissible by checking the four conditions in Definition 1. Furthermore, V solves the HJB equation (34) and shows quadratic growth. Therefore, by Lemma 5, we have the following conclusion.

Proposition 2. *For the unconstrained MV problem (32), the optimal value function $J^*(t,x) = V(t,x)$ and the optimal stochastic policy $\boldsymbol{\pi}^* = \boldsymbol{\pi}_c$.*

When there is no jump, we have $\sigma_J^2 = 0$ and thus recover the expressions of the optimal value function and optimal policy derived in Wang and Zhou (2020) for the unconstrained MV problem in the pure diffusion setting.

5.2 Parametrizations for learning

It is important to observe that the optimal value function, optimal policy and the Hamiltonian given by (36) take the same *structural* forms regardless of the presence of jumps, while the only differences are the constant coefficients in those functions. However, those coefficients are unknown anyway and will be parameterized in the implementation of our RL algorithms. Consequently, we can use the same parameterizations for the optimal value function and optimal q-function for learning as in the diffusion setting Jia and Zhou (2023). This important insight, concluded only *after* a rigorous theoretical analysis, shows that the continuous-time RL algorithms are robust to the presence of jumps and essentially model-free, at least for the MV problem.

Following Jia and Zhou (2023), we parametrize the value function as

$$
J^{\psi}(t,x;\omega) = (x-\omega)^2 e^{-\psi_3(T-t)} + \psi_2\left(t^2-T^2\right) + \psi_1(t-T) - (\omega-z)^2,
$$

and the q-function as

$$
q^{\phi}(t,x,a;w) = -\frac{e^{-\phi_2-\phi_3(T-t)}}{2}\left(a+\phi_1(x-w)\right)^2 - \frac{\theta}{2}\left[\log 2\pi\theta + \phi_2 + \phi_3(T-t)\right].
$$

Let $\psi = (\psi_1,\psi_2,\psi_3)^\top$ and $\phi = (\phi_1,\phi_2,\phi_3)^\top$. The policy associated with the parametric q-function is $\boldsymbol{\pi}^{\phi}(\cdot \mid t,x;w) = \mathcal{N}\left(-\phi_1(x-w), \theta e^{\phi_2+\phi_3(T-t)}\right)$. In addition to ψ and ϕ, we learn the Lagrange multiplier ω in the same way as in Jia and Zhou (2023) by the stochastic approximation algorithm that updates ω with a learning rate after a fixed number of iterations.

5.3 Simulation study

In our simulation experiment we use the same basic setting as in Jia and Zhou (2023): $x_0 = 1$, $z = 1.4$, $T = 1$ year, $\Delta t = 1/252$ years (corresponding to one trading day), and a chosen temperature parameter $\theta = 0.1$. We consider two market simulators: one is given by the Black–Scholes (BS) model and the other is Merton's jump-diffusion (MJD) model in which the Lévy density is a scaled Gaussian density, i.e.,

$$\nu(z) = \lambda \frac{1}{\sqrt{2\pi\delta^2}} \exp\left(-\frac{(z-m)^2}{2\delta^2}\right), \ \lambda > 0, \delta > 0, m \in \mathbb{R},$$

where λ is the arrival rate of the Poisson jumps. Under the latter model, we have

$$\sigma_J^2 = \lambda \left[\exp\left(2m + 2\delta^2\right) - 2\exp\left(m + \frac{1}{2}\delta^2\right) + 1\right].$$

To mimic the real market, we set the parameters of these two simulators by estimating them from the daily data of the S&P 500 index using maximum likelihood estimation. Our estimation data cover a long period from the beginning of 2000 to the end of 2023. In Table 1, we summarize the estimated parameter values (used for the simulators) and the corresponding value of ϕ_1^* in the optimal policy. Note that although we use a stochastic policy to interact with the environment during training to update the policy parameters, for actual execution of portfolio selection we apply a deterministic policy which is the *mean* part of the optimal stochastic policy after it has been learned. So it is *off-policy* learning here. The advantage of doing so among others is to reduce the variance of the final wealth; see Huang et al. (2022) for a discussion on this approach. As a result, here we only display the values for ϕ_1^* in these two environments and use them as benchmarks to check the convergence of our algorithm (see Figure 1).

Simulator	**Parameters**	**Optimal**
BS	$\mu = 0.0690, \sigma = 0.1965$	$\phi_1^* = 1.5940$
MJD	$\mu = 0.0636, \sigma = 0.1347, \lambda = 28.4910, m = -0.0039, \delta = 0.0275$	$\phi_1^* = 1.7869$

Table 1: Parameters used in the two simulators. The column "Optimal" reports the values of ϕ_1 in the optimal policies calculated using the respective simulators.

For offline learning, the Lagrange multiplier ω is updated after every $m = 10$ iterations, and the parameter vectors ψ and ϕ are initialized as zero vectors. The learning rates are set to be $\alpha_w = 0.05$, $\alpha_\psi = 0.001$, and $\alpha_\phi = 0.1$ with decay rate $l(j) = j^{-0.51}$. In each iteration, we generate 32 independent T-year trajectories to update the parameters. We train the model for $N = 2 \times 10^4$ iterations.

We also consider online learning with Δt equal to one trading day. We select a batch size of 128 trading days and update the parameters after this number of observations coming in. We set $m = 1$ for updating ω and initialize ψ and ϕ as zero vectors. The learning rates are set as $\alpha_w = 0.01$, $\alpha_\psi = 0.001$, and $\alpha_\phi = 0.05$ with decay rate $l(j) = j^{-0.51}$. Some of the rates are notably smaller under online learning because now we update with fewer observations and thus must be more cautious. The model is again trained for $N = 2 \times 10^4$ iterations.

Figure 1 plots the convergence behavior of offline and online learning under both simulators or market environments (one with jumps and one without). The algorithms have converged after a

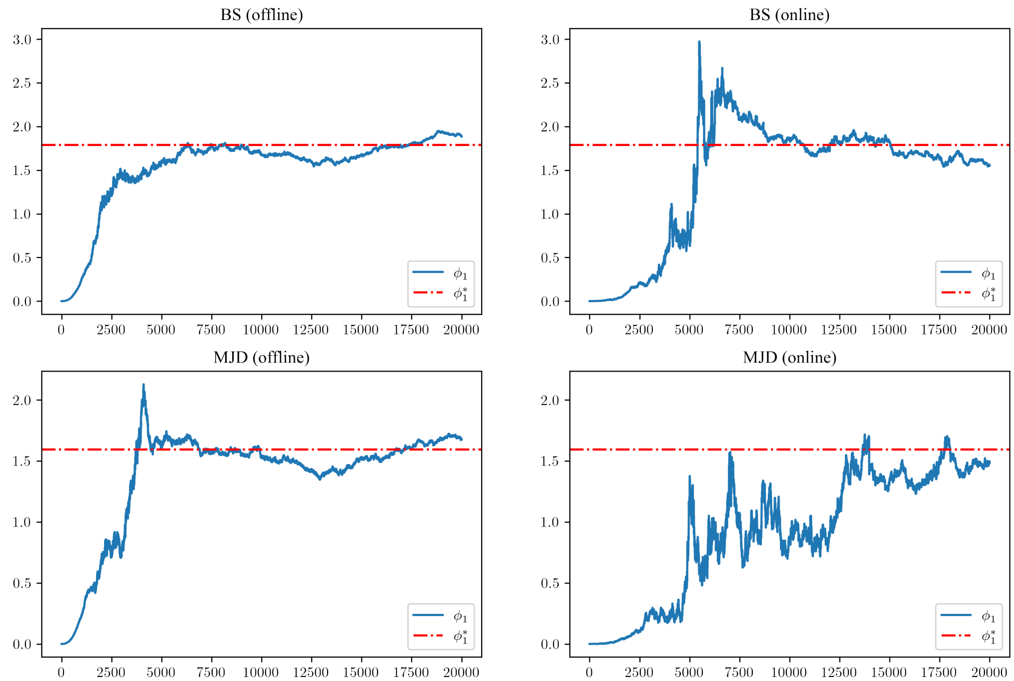

Figure 1: Convergence of the offline and online q-Learning algorithms under two market simulators for the policy parameter ϕ_1. The x- and y-axis show the iteration index and learned ϕ_1, respectively.

sufficient number of iterations, whether jumps are present or not. This demonstrates that convergence of the offline and online q-learning algorithms proposed in Jia and Zhou (2023) under the diffusion setting is robust to the presence of jumps for mean–variance portfolio selection. However, jumps in the environment can introduce more variability in the convergence process as seen from the plots.

6 Effects of Jumps

The theoretical analysis in the previous section shows that, *for mean–variance problem*, one does not need to know in advance whether or not the stock prices have jumps in order to carry out the RL task, because optimal stochastic policies are Gaussian and the corresponding value function and q-function have the same structures for parameterization irrespective of the presence of jumps. However, we stress that this is rather an exception than a rule. Here we give a counterexample.

Consider a modification of the mean–variance problem where the controlled system dynamic is

$$dX_t^a = a_t\sigma\rho dt + a_t\sigma dB_t + \int_{\mathbb{R}} \gamma(a_t, z)\widetilde{N}(dt, dz), \tag{37}$$

with

$$\gamma(a,z) = a^2, \tag{38}$$

and the exploratory objective is

$$J^*(t,x;w) = \min_{\boldsymbol{\pi}} \mathbb{E}^{\bar{\mathbb{P}}}_{t,x}\left[(X^{\boldsymbol{\pi}}_T - w)^2 + \theta\int_t^T \log\boldsymbol{\pi}(a^{\boldsymbol{\pi}}_s|s,X^{\boldsymbol{\pi}}_{s-})ds\right] - (\omega - z)^2.$$

Note that this is *not* a mean–variance portfolio selection problem because (37) does not correspond to a self-financed wealth equation with a reasonably modelled stock price process.

The Hamiltonian is given by

$$\begin{aligned} H(t,x,a,v_x,v_{xx},v) = a\sigma\rho v_x(t,x) + \frac{1}{2}a^2\sigma^2 v_{xx}(t,x) \\ + \int_{\mathbb{R}}\left(v(t,x+a^2) - v(t,x) - a^2\cdot v_x(t,x)\right)\nu(dz). \end{aligned} \tag{39}$$

If an optimal stochastic policy exists, then it must be

$$\boldsymbol{\pi}^*(a|t,x) \propto \exp\left(-\frac{1}{\theta}H(t,x,a,J^*_x,J^*_{xx},J^*)\right). \tag{40}$$

We show by contradiction that the optimal stochastic policy can not be Gaussian in this case. Note that if there is no optimal stochastic policy, then it would already demonstrate that jumps matter because the optimal stochastic policy for the case of no jumps exists and is Gaussian.

Remark 4. *The existence of optimal stochastic policy in (40) is equivalent to the integrability of the quantity* $\exp\left(\frac{1}{\theta}H(t,x,a,J^*_x,J^*_{xx},J^*)\right)$ *over* $a\in\mathcal{A} = (-\infty,\infty)$*. This integrability depends on the tail behavior of the Hamiltonian and, in particular, the behavior of* $J^*(t,x+a^2)$ *when* a^2 *is large.*

Suppose the optimal stochastic policy $\boldsymbol{\pi}^*(\cdot|t,x)$ is Gaussian for all (t,x), implying that the Hamiltonian $H(t,x,a,J^*_x,J^*_{xx},J^*)$ is a quadratic function of a. It then follows from (39) that there exist functions $h_1(t,x)$ and $h_2(t,x)$ such that

$$J^*(t,x+a^2) - J^*(t,x) - a^2 J^*_x(t,x) = a^2\cdot h_1(t,x) + a\cdot h_2(t,x), \quad \text{for all } (t,x,a). \tag{41}$$

We do not put a term independent of a on the right-hand side because the left-hand side is zero when $a=0$. Taking derivative with respect to a, we obtain

$$J^*_x(t,x+a^2)\cdot 2a - 2aJ^*_x(t,x) = 2a\cdot h_1(t,x) + h_2(t,x), \quad \text{for all } (t,x,a).$$

Setting $a=0$, we get $h_2 = 0$. It follows that

$$a\cdot\left[J^*_x(t,x+a^2) - J^*_x(t,x) - h_1(t,x)\right] = 0 \quad \text{for all } (t,x,a).$$

Hence we have $h_1(t,x) = J^*_x(t,x+a^2) - J^*_x(t,x)$ for any $a\neq 0$. Sending a to zero yields $h_1(t,x) = 0$ for all (t,x). Therefore, we obtain from (41) that

$$J^*(t,x+a^2) - J^*(t,x) - a^2 J^*_x(t,x) = 0, \quad \text{for all } (t,x,a).$$

Taking derivative in a in the above we have

$$J_x^*(t, x + a^2) - J_x^*(t, x) = 0, \quad \text{for all } (t, x, a).$$

Thus J_x^* is constant in x or J^* is affine in x, leading to $J^*(t,x) = g_1(t)x + g_2(t)$ for some functions $g_1(t)$ and $g_2(t)$. The resulting Hamiltonian becomes

$$H(t, x, a, J_x^*, J_{xx}^*, J^*) = a\sigma\rho g_1(t).$$

This is linear in $a \in \mathcal{A} = (-\infty, \infty)$ and hence the integral $\int_{\mathcal{A}} \exp\left(-\frac{1}{\theta} H(t, x, a, J_x^*, J_{xx}^*, J^*)\right) da$ does not exist. It follows that $\boldsymbol{\pi}^*(\cdot|t,x)$ does not exist, which is a contradiction. Therefore, we have shown that under (38), the optimal stochastic policy either does not exist or is not Gaussian when it exists.

Remark 5. *The argument above works for $\gamma(a,z) = a^m$ for any $m > 1$.*

7 Conclusions

Fluctuations in data or time series coming from nonlinear, complex dynamic systems are characterized by two types: slow changes and sudden jumps, the latter occurring much rarely than the former. Hence jump-diffusions capture the key *structural* characteristics of many data generating processes in areas such as physics, astrophysics, earth science, engineering, finance, and medicine. As a result, RL for jump-diffusions is important both theoretically and practically. This paper endeavors to lay a theoretical foundation for the study. A key insight from this research is that temporal–difference algorithms designed for diffusions can work seamlessly for jump-diffusions. However, unless using general neural networks, policy parameterization does need to respond to the presence of jumps if one is to take advantage of any special structure of an underlying problem.

There are plenty of open questions starting from here, including the convergence of the algorithms, regret bounds, decaying rates of the temperature parameters, and learning rates of the gradient decent and/or stochastic approximation procedures involved.

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A Proofs

Proof of Lemma 2. We observe that for each $k = 1, \cdots, \ell$,

$$\int_{\mathbb{R}\times[0,1]^n} |\gamma_k(t, x, G^{\boldsymbol{\pi}_{t,x}}(u), z)|^p \nu_k(dz)du = \int_{\mathcal{A}} \int_{\mathbb{R}} |\gamma_k(t, x, a, z)|^p \nu_k(dz)\boldsymbol{\pi}(a|t, x)da.$$

From Assumption 1-(iii), we have $\sum_{k=1}^{\ell} \int_{\mathbb{R}} |\gamma_k(t, x, a, z)|^p \nu_k(dz) \leq C_p(1 + |x|^p)$ for any $(t, x, a) \in [0, T] \times \mathbb{R}^d \times \mathcal{A}$, where C_p does not depend on a. Thus, integrating over a with the measure $\boldsymbol{\pi}(a|t, x)da$ preserves the linear growth property. $\square$

Proof of Lemma 3. We consider

$$\int_{\mathbb{R}\times[0,1]^n} \left|\gamma_k(t, x, G^{\boldsymbol{\pi}_{t,x}}(u), z) - \gamma_k\left(t, x', G^{\boldsymbol{\pi}_{t,x'}}(u), z\right)\right|^p \nu_k(dz)du,$$

which is bounded by

$$\begin{aligned}
C_p \Bigg(& \int_{\mathbb{R}\times[0,1]^n} |\gamma_k(t, x, G^{\boldsymbol{\pi}_{t,x}}(u), z) - \gamma_k(t, x, G^{\boldsymbol{\pi}_{t,x'}}(u), z)|^p \nu_k(dz)du \\
& + \int_{\mathbb{R}\times[0,1]^n} \left|\gamma_k(t, x, G^{\boldsymbol{\pi}_{t,x'}}(u), z) - \gamma_k\left(t, x', G^{\boldsymbol{\pi}_{t,x'}}(u), z\right)\right|^p \nu_k(dz)du \Bigg)
\end{aligned}$$

for some constant $C_p > 0$.

For the first integral, using Assumption 2 we obtain

$$\begin{aligned}
& \int_{\mathbb{R}\times[0,1]^n} |\gamma_k(t, x, G^{\boldsymbol{\pi}_{t,x}}(u), z) - \gamma_k(t, x, G^{\boldsymbol{\pi}_{t,x'}}(u), z)|^p \nu_k(dz)du \\
\leq & C_{K,p} \int_{[0,1]^n} \left|G^{\boldsymbol{\pi}}(t, x, u) - G^{\boldsymbol{\pi}}(t, x', u)\right|^p du \leq C'_{K,p}|x - x'|^p.
\end{aligned}$$

For the second integral, we have

$$\begin{aligned}
& \int_{\mathbb{R}\times[0,1]^n} \left|\gamma_k(t, x, G^{\boldsymbol{\pi}_{t,x'}}(u), z) - \gamma_k\left(t, x', G^{\boldsymbol{\pi}_{t,x'}}(u), z\right)\right|^p \nu_k(dz)du \\
= & \int_{\mathcal{A}} \int_{\mathbb{R}} \left|\gamma_k(t, x, a, z) - \gamma_k\left(t, x', a, z\right)\right|^p \nu_k(dz)\boldsymbol{\pi}(a|t, x')da \\
\leq & \int_{\mathcal{A}} C''_{K,p}|x - x'|^p \boldsymbol{\pi}(a|t, x')da \\
= & C''_{K,p}|x - x'|^p,
\end{aligned}$$

where we used Assumption 1-(ii) and $C''_{K,p}$ is the constant there. The desired claim is obtained by combining these results. $\square$

Proof of Lemma 4. Fix $t \in [0, T)$ and suppose $\tilde{X}_t^{\pi} = x$. Define a sequence of stopping times $\tau_n = \inf\{s \geq t : |\tilde{X}_s^{\pi}| \geq n\}$ for $n \in \mathbb{N}$. Applying Itô's formula (5) to the process $e^{-\beta s}\phi(s, \tilde{X}_s^{\pi})$, where $\tilde{X}_s^{\pi}$ follows the exploratory SDE (17), we obtain $\forall t' \in [t, T]$,

$$e^{-\beta(t'\wedge\tau_n)}\phi(t' \wedge \tau_n, \tilde{X}_{r\wedge\tau_n}^{\pi}) - e^{-\beta t}\phi(t, x)$$

$$
\begin{aligned}
&= \int_t^{t'\wedge\tau_n} e^{-\beta s}\left(\mathcal{L}^{\boldsymbol{\pi}}\phi(s,\tilde{X}^{\pi}_{s-}) - \beta\phi(s,\tilde{X}^{\pi}_{s-})\right) ds \\
&+ \int_t^{t'\wedge\tau_n} e^{-\beta s}\phi_x(s,\tilde{X}^{\pi}_{s-})\circ\tilde{\sigma}(s,\tilde{X}^{\pi}_{s-},\pi(\cdot|s,\tilde{X}^{\pi}_{s-}))dW_s \qquad (42)\\
&+ \int_t^{t'\wedge\tau_n} e^{-\beta s}\sum_{k=1}^{\ell}\int_{\mathbb{R}\times[0,1]^n}\left(\phi(s,\tilde{X}^{\pi}_{s-}+\gamma_k(s,\tilde{X}^{\pi}_{s-},G_{\boldsymbol{\pi}}(s,\tilde{X}^{\pi}_{s-},u),z)) - \phi(s,\tilde{X}^{\pi}_{s-})\right)\widetilde{N}'_k(ds,dz,du), \qquad (43)
\end{aligned}
$$

where $\mathcal{L}^{\boldsymbol{\pi}}$ is the generator of the exploratory process $(\tilde{X}^{\pi}_t)$ given in (16). We next show that the expectations of (42) and (43) are zero.

Note that for $s\in[t,t'\wedge\tau_n]$, $|\tilde{X}^{\pi}_{s-}|\le n$. Thus, $\phi_x(s,\tilde{X}^{\pi}_{s-})$ is also bounded. Using the linear growth of $\tilde{\sigma}(s,\tilde{X}^{\pi}_{s-},\pi(\cdot|s,\tilde{X}^{\pi}_{s-}))$ in Lemma 1 and the moment estimate (18), we can see that (42) is a square-integrable martingale and hence its expectation is zero.

Next, we analyze the following stochastic integral:

$$
\int_t^{t'\wedge\tau_n} e^{-\beta s}\int_{\mathbb{R}\times[0,1]^n}\left(\phi(s,\tilde{X}^{\pi}_{s-}+\gamma_k(s,\tilde{X}^{\pi}_{s-},G_{\boldsymbol{\pi}}(s,\tilde{X}^{\pi}_{s-},u),z)) - \phi(s,\tilde{X}^{\pi}_{s-})\right)\widetilde{N}_k(ds,dz,du). \qquad (44)
$$

We consider the finite and infinite jump activity cases separately.

Case 1: $\int_{\mathbb{R}}\nu_k(dz)<\infty$. In this case, both of the processes

$$
\int_t^{t'\wedge\tau_n} e^{-\beta s}\int_{\mathbb{R}\times[0,1]^n}\phi(s,\tilde{X}^{\pi}_{s-}+\gamma_k(s,\tilde{X}^{\pi}_{s-},G_{\boldsymbol{\pi}}(s,\tilde{X}^{\pi}_{s-},u),z))\widetilde{N}_k(ds,dz,du), \qquad (45)
$$

$$
\int_t^{t'\wedge\tau_n} e^{-\beta s}\int_{\mathbb{R}\times[0,1]^n}\phi(s,\tilde{X}^{\pi}_{s-})\widetilde{N}_k(ds,dz,du), \qquad (46)
$$

are square-integrable martingales and hence have zero expectations. We prove this claim for (45) and analyzing (46) is entirely analogous.

Using the polynomial growth of ϕ, Lemma 2, and $|\tilde{X}^{\pi}_{s-}|\le n$ for $s\in[t,t'\wedge\tau_n]$, we obtain

$$
\begin{aligned}
&\mathbb{E}^{\bar{\mathbb{P}}}_{t,x}\left[\int_t^{t'\wedge\tau_n} e^{-2\beta s}\int_{\mathbb{R}\times[0,1]^n}\left|\phi(s,\tilde{X}^{\pi}_{s-}+\gamma_k(s,\tilde{X}^{\pi}_{s-},G_{\boldsymbol{\pi}}(s,\tilde{X}^{\pi}_{s-},u),z))\right|^2\nu_k(dz)duds\right]\\
\le & C_p\cdot\mathbb{E}^{\bar{\mathbb{P}}}_{t,x}\left[\int_t^{t'\wedge\tau_n}\int_{\mathbb{R}\times[0,1]^n}\left(1+|\tilde{X}^{\pi}_{s-}|^p+|\gamma_k(s,\tilde{X}^{\pi}_{s-},G_{\boldsymbol{\pi}}(s,\tilde{X}^{\pi}_{s-},u),z))|^p\right)\nu_k(dz)duds\right]\\
\le & C'_p\cdot\mathbb{E}^{\bar{\mathbb{P}}}_{t,x}\left[\int_t^{t'\wedge\tau_n}(1+|\tilde{X}^{\pi}_{s-}|^p)ds\right]<\infty.
\end{aligned}
$$

This implies the process (45) is a square-integrable martingale; see e.g., (Situ 2006, Section 1.9) for square-integrability of stochastic integrals with respect to compensated Poisson random measures.

Case 2: $\int_{\mathbb{R}}\nu_k(dz)=\infty$. Let $B_1=\{|z|\le 1\}\times[0,1]^n$ and $B_1^c=\{|z|>1\}\times[0,1]^n$. The stochastic integral (44) can be written as the sum of two integrals:

$$
\int_t^{t'\wedge\tau_n} e^{-\beta s}\int_{B_1}\left(\phi(s,\tilde{X}^{\pi}_{s-}+\gamma_k(s,\tilde{X}^{\pi}_{s-},G_{\boldsymbol{\pi}}(s,\tilde{X}^{\pi}_{s-},u),z)) - \phi(s,\tilde{X}^{\pi}_{s-})\right)\widetilde{N}_k(ds,dz,du) \qquad (47)
$$

$$+\int_t^{t'\wedge\tau_n} e^{-\beta s}\int_{B_1^c}\Big(\phi(s,\tilde{X}^\pi_{s-}+\gamma_k(s,\tilde{X}^\pi_{s-},G_{\boldsymbol{\pi}}(s,\tilde{X}^\pi_{s-},u),z))-\phi(s,\tilde{X}^\pi_{s-})\Big)\widetilde{N}_k(ds,dz,du). \quad (48)$$

Using the mean-value theorem, the stochastic integral (47) is equal to

$$\int_t^{t'\wedge\tau_n} e^{-\beta s}\int_{B_1}\phi_x(s,\tilde{X}^\pi_{s-}+\alpha_s\gamma_k(s,\tilde{X}^\pi_{s-},G_{\boldsymbol{\pi}}(s,\tilde{X}^\pi_{s-},u),z))\circ\gamma_k(s,\tilde{X}^\pi_{s-},G_{\boldsymbol{\pi}}(s,\tilde{X}^\pi_{s-},u),z)\widetilde{N}_k(ds,dz,du)$$

for some $\alpha_s\in[0,1]$. For $s\in[t,t'\wedge\tau_n]$, $|\tilde{X}^\pi_{s-}|\le n$. By Assumption 1′-(v), $|\gamma_k(s,\tilde{X}^\pi_{s-},G_{\boldsymbol{\pi}}(s,\tilde{X}^\pi_{s-},u),z)|$ is bounded for any $|z|\le 1$, $s\in[t,t'\wedge\tau_n]$ and $u\in[0,1]^n$, which further implies the boundedness of $|\phi_x(s,\tilde{X}^\pi_{s-}+\alpha_s\gamma_k(s,\tilde{X}^\pi_{s-},G_{\boldsymbol{\pi}}(s,\tilde{X}^\pi_{s-},u),z))|$. Now, for each $j=1,\cdots,d$,

$$\int_t^{t'\wedge\tau_n} e^{-\beta s}\int_{B_1}\phi_{x_j}(s,\tilde{X}^\pi_{s-}+\alpha_s\gamma_k(s,\tilde{X}^\pi_{s-},G_{\boldsymbol{\pi}}(s,\tilde{X}^\pi_{s-},u),z))\gamma_{jk}(s,\tilde{X}^\pi_{s-},G_{\boldsymbol{\pi}}(s,\tilde{X}^\pi_{s-},u),z)\widetilde{N}_k(ds,dz,du)$$

is a square-integrable martingale because

$$\begin{aligned}
&\mathbb{E}^{\bar{\mathbb{P}}}_{t,x}\left[\int_t^{t'\wedge\tau_n} e^{-2\beta s}\int_{B_1}\phi^2_{x_j}(s,\tilde{X}^\pi_{s-}+\alpha_s\gamma_k(s,\tilde{X}^\pi_{s-},G_{\boldsymbol{\pi}}(s,\tilde{X}^\pi_{s-},u),z))\gamma^2_{jk}(s,\tilde{X}^\pi_{s-},G_{\boldsymbol{\pi}}(s,\tilde{X}^\pi_{s-},u),z)\nu_k(dz)duds\right]\\
\le& C\cdot\mathbb{E}^{\bar{\mathbb{P}}}_{t,x}\left[\int_t^{t'\wedge\tau_n}\int_{B_1}\gamma^2_{jk}(s,\tilde{X}^\pi_{s-},G_{\boldsymbol{\pi}}(s,\tilde{X}^\pi_{s-},u),z)\nu_k(dz)duds\right]\\
\le& C'\cdot\mathbb{E}^{\bar{\mathbb{P}}}_{t,x}\left[\int_t^{t'\wedge\tau_n}(1+|\tilde{X}^\pi_{s-}|^2)ds\right]<\infty,
\end{aligned}$$

where we used Lemma 2 and boundedness of $|\tilde{X}^\pi_{s-}|$ in the above. Thus, (47) is a square-integrable martingale with mean zero.

For (48), we can use the same arguments as in the finite activity case by observing $\int_{|z|>1}\nu_k(dz)<\infty$ to show that each of the two processes in (48) is a square-integrable martingale with mean zero.

Combining the results above, setting $t'=T$, and taking expectation, we obtain

$$\mathbb{E}^{\bar{\mathbb{P}}}_{t,x}\left[e^{-\beta(T\wedge\tau_n)}\phi(\tilde{X}^{\boldsymbol{\pi}}_{T\wedge\tau_n})\right]-e^{-\beta t}\phi(t,x)=\mathbb{E}^{\bar{\mathbb{P}}}_{t,x}\left[\int_t^{T\wedge\tau_n}e^{-\beta s}\Big(\mathcal{L}^{\boldsymbol{\pi}}\phi(s,\tilde{X}^\pi_{s-})-\beta\phi(s,\tilde{X}^\pi_{s-})\Big)ds\right].$$

As $\phi(s,x)$ satisfies Equation (21), it follows from (7) that

$$\begin{aligned}
\phi(t,x)=\mathbb{E}^{\bar{\mathbb{P}}}_{t,x}\Bigg[&\int_t^{T\wedge\tau_n}e^{-\beta(s-t)}\int_{\mathcal{A}}\Big(r(s,\tilde{X}^{\boldsymbol{\pi}}_{s-},a)-\theta\log\boldsymbol{\pi}(a|s,\tilde{X}^\pi_{s-})\Big)\boldsymbol{\pi}(a|s,\tilde{X}^{\boldsymbol{\pi}}_{s-})dads\\
&+e^{-\beta(T\wedge\tau_n-t)}\phi(\tilde{X}^{\boldsymbol{\pi}}_{T\wedge\tau_n})\Bigg]. \qquad (49)
\end{aligned}$$

It follows from Assumption 1-(iv), Definition 1-(iii), and Equation (22) that the term on the right-hand side of (49) is dominated by $C(1+\sup_{t\le s\le T}|\tilde{X}^{\boldsymbol{\pi}}_s|^p)$ for some $p\ge 2$, which has finite expectation from the moment estimate (18). Therefore, sending n to infinity in (49) and applying the dominated convergence theorem, we obtain $\phi(t,x)=J(t,x,\boldsymbol{\pi})$. $\square$

Proof of Lemma 5. We first maximize the integral in (23). Applying (Jia and Zhou 2023, Lemma 13), we deduce that $\boldsymbol{\pi}^*$ given by (24) is the unique maximizer. Next, we show that $\psi(t,x)$ is the optimal value function.

On one hand, given any admissible stochastic policy $\boldsymbol{\pi} \in \boldsymbol{\Pi}$, from (23) we have

$$\frac{\partial \psi(t,x)}{\partial t} + \int_{\mathcal{A}} \{H(t,x,a,\psi_x,\psi_{xx},\psi) - \theta \log \boldsymbol{\pi}(a|t,x)\}\boldsymbol{\pi}(a|t,x)da - \beta\psi(t,x) \leq 0.$$

Using similar arguments as in the proof of Lemma 4, we obtain $J(t,x,\boldsymbol{\pi}) \leq \psi(t,x)$ for any $\boldsymbol{\pi} \in \boldsymbol{\Pi}$. Thus, $J^*(t,x) \leq \psi(t,x)$.

On the other hand, Equation (23) becomes

$$\frac{\partial \psi(t,x)}{\partial t} + \int_{\mathcal{A}} \{H(t,x,a,\psi_x,\psi_{xx},\psi) - \theta \log \boldsymbol{\pi}^*(a)\}\boldsymbol{\pi}^*(a)da - \beta\psi(t,x) = 0,$$

with $\psi(T,x) = h(x)$. By Lemma 4, we obtain that $J(t,x,\boldsymbol{\pi}^*) = \psi(t,x)$. It follows that $J^*(t,x) \geq \psi(t,x)$.

Combining these results, we conclude that $J^*(t,x) = \psi(t,x)$ and $\boldsymbol{\pi}^*$ is the optimal stochastic policy. $\square$

Proof of Theorem 2. Consider part (i). Applying Itô's formula to the value function of policy $\boldsymbol{\pi}$ over the sample state process defined by (8) and using the definition of q-function, we obtain that for $0 \leq t < s \leq T$,

$$\begin{aligned}
&e^{-\beta s} J(s, X_s^{\boldsymbol{\pi}}; \boldsymbol{\pi}) - e^{-\beta t} J(t,x;\boldsymbol{\pi}) + \int_t^s e^{-\beta\tau}[r(\tau, X_{\tau-}^{\boldsymbol{\pi}}, a_\tau^{\boldsymbol{\pi}}) - \hat{q}(\tau, X_{\tau-}^{\boldsymbol{\pi}}, a_\tau^{\boldsymbol{\pi}})]d\tau \\
&= \int_t^s e^{-\beta\tau}[q(\tau, X_{\tau-}^{\boldsymbol{\pi}}, a_\tau^{\boldsymbol{\pi}}; \boldsymbol{\pi}) - \hat{q}(\tau, X_{\tau-}^{\boldsymbol{\pi}}, a_\tau^{\boldsymbol{\pi}})]d\tau + \int_t^s e^{-\beta\tau} J_x(\tau, X_{\tau-}^{\boldsymbol{\pi}}; \boldsymbol{\pi}) \circ \sigma(\tau, X_{\tau-}^{\boldsymbol{\pi}}, a_\tau^{\boldsymbol{\pi}}) dW_\tau \\
&\quad + \sum_{k=1}^{\ell} \int_t^s e^{-\beta\tau} \int_{\mathbb{R}} \left(J(\tau, X_{\tau-}^{\boldsymbol{\pi}} + \gamma_k(\tau, X_{\tau-}^{\boldsymbol{\pi}}, a_\tau^{\boldsymbol{\pi}}, z)) - J(\tau, X_{\tau-}^{\boldsymbol{\pi}}; \boldsymbol{\pi})\right) \widetilde{N}_k(d\tau, dz).
\end{aligned} \tag{50}$$

Suppose $\hat{q}(t,x,a) = q(t,x,a;\boldsymbol{\pi})$ for all (t,x,a). Hence the first term on the right-hand side of (50) is zero. We verify the following two conditions:

$$\mathbb{E}_{t,x}^{\bar{\mathbb{P}}}\left[\int_t^T e^{-2\beta\tau} |J_x(\tau, X_{\tau-}^{\boldsymbol{\pi}}; \boldsymbol{\pi}) \circ \sigma(\tau, X_{\tau-}^{\pi}, a_\tau^{\pi})|^2 d\tau\right] < \infty, \tag{51}$$

$$\mathbb{E}_{t,x}^{\bar{\mathbb{P}}}\left[\int_t^T e^{-2\beta\tau} \int_{\mathbb{R}} \left|J(\tau, X_{\tau-}^{\boldsymbol{\pi}} + \gamma_k(\tau, X_{\tau-}^{\boldsymbol{\pi}}, a_\tau^{\boldsymbol{\pi}}, z); \boldsymbol{\pi}) - J(\tau, X_{\tau-}^{\boldsymbol{\pi}}; \boldsymbol{\pi})\right|^2 \nu_k(dz) d\tau\right] < \infty. \tag{52}$$

Equation (51) follows from Assumption 1-(iii), the polynomial growth of J_x in x, and the moment estimate (19). For (52), we apply the mean-value theorem and the integral becomes

$$\int_t^T e^{-2\beta\tau} \int_{\mathbb{R}} \left|J_x(\tau, X_{\tau-}^{\boldsymbol{\pi}} + \alpha_\tau \gamma_k(\tau, X_{\tau-}^{\boldsymbol{\pi}}, a_\tau^{\boldsymbol{\pi}}, z); \boldsymbol{\pi}) \circ \gamma_k(\tau, X_{\tau-}^{\boldsymbol{\pi}}, a_\tau^{\boldsymbol{\pi}}, z)\right|^2 \nu_k(dz) d\tau$$

for some $\alpha_\tau \in [0,1]$. Using the polynomial growth of J_x in x, we can bound the integral by

$$\int_t^T e^{-2\beta\tau} \int_{\mathbb{R}} \left|J_x(\tau, X_{\tau-}^{\boldsymbol{\pi}} + \alpha_\tau \gamma_k(\tau, X_{\tau-}^{\boldsymbol{\pi}}, a_\tau^{\boldsymbol{\pi}}, z); \boldsymbol{\pi})\right|^2 \cdot \left|\gamma_k(\tau, X_{\tau-}^{\boldsymbol{\pi}}, a_\tau^{\boldsymbol{\pi}}, z)\right|^2 \nu_k(dz) d\tau$$

$$
\begin{aligned}
&\leq C\int_t^T e^{-2\beta\tau}\int_{\mathbb{R}}\left(1+|X^{\boldsymbol{\pi}}_{\tau-}+\alpha_\tau\gamma_k(\tau,X^{\boldsymbol{\pi}}_{\tau-},a^{\boldsymbol{\pi}}_\tau,z)|^p\right)^2\cdot\left|\gamma_k(\tau,X^{\boldsymbol{\pi}}_{\tau-},a^{\boldsymbol{\pi}}_\tau,z)\right|^2\nu_k(dz)d\tau\\
&\leq C'\int_t^T e^{-2\beta\tau}\int_{\mathbb{R}}\left(1+\left|X^{\boldsymbol{\pi}}_{\tau-}\right|^p+\left|\gamma_k(\tau,X^{\boldsymbol{\pi}}_{\tau-},a^{\boldsymbol{\pi}}_\tau,z)\right|^p\right)^2\cdot\left|\gamma_k(\tau,X^{\boldsymbol{\pi}}_{\tau-},a^{\boldsymbol{\pi}}_\tau,z)\right|^2\nu_k(dz)d\tau\\
&\leq C'\int_t^T\Bigg((1+\left|X^{\boldsymbol{\pi}}_{\tau-}\right|^p)^2\int_{\mathbb{R}}\left|\gamma_k(\tau,X^{\boldsymbol{\pi}}_{\tau-},a^{\boldsymbol{\pi}}_\tau,z)\right|^2\nu_k(dz)\\
&\quad+2(1+\left|X^{\boldsymbol{\pi}}_{\tau-}\right|^p)\int_{\mathbb{R}}\left|\gamma_k(\tau,X^{\boldsymbol{\pi}}_{\tau-},a^{\boldsymbol{\pi}}_\tau,z)\right|^{p+2}\nu_k(dz)+\int_{\mathbb{R}}\left|\gamma_k(\tau,X^{\boldsymbol{\pi}}_{\tau-},a^{\boldsymbol{\pi}}_\tau,z)\right|^{2p+2}\nu_k(dz)\Bigg)d\tau
\end{aligned}
$$

Using Assumption 1-(iii) and the moment estimate (19), we obtain (52). It follows that the second and third processes on the right-hand side of (50) are $(\{\mathcal{G}_s\}_{s\geq 0},\bar{\mathbb{P}})$-martingales and thus we have the martingale property of the process given by (25).

Conversely, if (25) is a $(\{\mathcal{G}_s\}_{s\geq 0},\bar{\mathbb{P}})$-martingale, we see from (50) that the process

$$
\int_t^s e^{-\beta\tau}[q(\tau,X^{\boldsymbol{\pi}}_{\tau-},a^{\boldsymbol{\pi}}_\tau;\boldsymbol{\pi})-\hat{q}(\tau,X^{\boldsymbol{\pi}}_{\tau-},a^{\boldsymbol{\pi}}_\tau)]d\tau
$$

is also a $(\{\mathcal{G}_s\}_{s\geq 0},\bar{\mathbb{P}})$-martingale. Furthermore, it has continuous sample paths and finite variation and thus is equal to zero $\bar{\mathbb{P}}$-almost surely. We can then follow the argument in the proof of Theorem 6 in Jia and Zhou (2023) to show that $\hat{q}(t,x,a)=q(t,x,a;\boldsymbol{\pi})$ for all (t,x,a). There is only one step in their proof that we need to modify due to the presence of jumps.

Specifically, consider the sample state process $X^{\boldsymbol{\pi}}$ starting from some time-state-action (t^*,x^*,a^*). Fix $\delta>0$ and define

$$
T_\delta=\inf\{t'\geq t^*:|X^{\boldsymbol{\pi}}_{t'}-x^*|>\delta\}\wedge(t^*+\delta).
$$

In the pure diffusion case, Jia and Zhou (2023) uses the continuity of the sample paths of $X^{\boldsymbol{\pi}}$ to argue that $T_\delta>t^*$, $\bar{\mathbb{P}}$-almost surely. This result still holds with presence of jumps, because the Lévy processes that drive our controlled state $X^{\boldsymbol{\pi}}$ are stochastic continuous, i.e., the probability of having a jump at the fixed time t^* is zero.

To prove parts (ii) and (iii), we can apply the arguments used in proving part (i) together with those arguments from the proof of Theorem 6 in Jia and Zhou (2023). The details are omitted. □