Internet Supplement
to
Hydrodynamic limit of order book dynamics

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Abstract

In this paper, we establish a fluid limit for a two–sided Markov order book model. Our main result states that in a certain asymptotic regime, a pair of measure-valued processes representing the "sell-side shape" and "buy-side shape" of an order book converges to a pair of deterministic measure-valued processes in a certain sense. We also test our fluid approximation on data. The empirical results suggest that the approximation is reasonably good for liquidly–traded stocks in certain time periods.

1 Overview

This is an internet supplement to the main paper [2]. It provides detailed empirical results and supplements Section 4 of the main paper.

2 Summary statistics and empirical results

In this section we present summary statistics for the representative example Bank of America (BAC) traded on NYSE Arca on August 5, 2010, and we summarize the empirical results on comparison of empirical and theoretical shapes of BAC during 12:45pm-13:05pm.

2.1 Summary statistics

In this section we present summary statistics for BAC traded on NYSE Arca on August 5, 2010.

We first present some general summary statistics. During 12:45pm-13:05pm, the total number of BAC order book events on NYSE Arca is 14841. The average limit sell order size is 363 shares, which we use as unit order size. Using time format **hh:mm:ss.milliseconds**, the first event during the 20-minute window occurs at 12:45:00.015. At this "initial" time, the best bid price is \$13.99 and the best ask is \$14.00. During this 20-minute window, the bid price changes are -1, +1, -1, +1 (in the unit of one cent/tick) at time instants 13:03:18.184, 13:03:19.599, 13:03:42.011, 13:04:21.472; the ask price changes are -1, +1, -1, +1, -1, +1 (in the unit of one cent/tick) at times 13:03:19.112, 13:03:19.597, 13:03:42.013,

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13:04:21.469, 13:04:21.470, 13:04:21.471. Hence the best bid ranges from \$13.98 to \$13.99 and the best ask ranges from \$13.99 to \$14.00. They barely change during this 20-minute time period, which is consistent with our result in part (a) of Theorem 3.1 in the main paper.

We next report in Table 1 the shape of the order book of BAC at 12:45pm, more precisely, the first snapshot of the order book after 12:45pm. This shape is obtained at 15 milliseconds after 12:45pm. From Table 1, we observe that for this initial shape, the number of limit orders on each price level is approximately on the order of $n = 100$ (in the unit of 363 shares). This is consistent with Assumption 2.1 in the main paper.

Table 1: Number of BAC limit orders on Arca at 12:45pm, August 5, 2010. The quantities are computed from dividing outstanding limit order volume (shares) on each price level by unit order size (363 shares). In the table, level i represents i ticks away from the opposite best quote.

Sell side										
Levels 1-10	136	184	208	194	149	156	179	94	118	102
Levels 11-20	83	66	73	63	66	91	52	50	40	231
Levels 21-30	14	19	8	9	6	61	1	6	20	43
Buy side										
Levels 1-10	223	517	256	165	202	142	93	126	141	202
Levels 11-20	107	75	389	22	102	1	3	21	14	374
Levels 21-30	10	61	34	15	97	35	7	5	10	34

We finally summarize in Tables 2-5 the order flow rates for each of the four time windows discussed in Section 4 of the main paper. The estimation procedure follows [1], but we allow for the dependence of order flow intensities on the buy/sell side. For illustration, we present results for price levels 10 ticks above the best bid and 10 ticks below the best ask where most order activities concentrate. There are seven columns in each of the Tables 2-5. The first column i signifies the number of price ticks away from the opposite best quote; the second column Λ_A^n represents limit sell order arrival rate (number of orders per second); the third column Θ_A^n is the limit sell order cancel rate (per second per order); the fourth column N_A^n records the number of sell-side events (including limit sell order arrivals, limit sell order cancellations and market buy order arrivals); The fifth to seventh columns correspond to limit buy order arrival rate, limit buy order cancellation rate, and the number of buy-side events. Market order arrival rate and the total number of events in each time period are recorded below each table.

2.2 Empirical results

In this section we summarize the empirical results on the comparisons of empirical and theoretical shapes of BAC during 12:45pm-13:05pm using Tables 6-9. For the clarity of the presentation, we focus on price levels 10 ticks above the best bid and 10 ticks below the best ask.

We now explain the entries in the tables and illustrate them using Table 6. For the sell side, each entry in the row "12:50 Arca" represents the total number of limit sell orders (in the unit of 363-shares) in each price bin at 12:50pm on NYSE Arca. Each quantity corresponds to left-hand side of (4.1) in the main paper. On the other hand, each entry in the row "12:50 Model" is computed from our limiting model φ^+ given in the main paper. Each number corresponds to right-hand side of (4.1). "Error" for each price bin is defined by (entry in row "12:50 Model")/(entry in row "12:50 Arca")-100%, which captures the accuracy of our approximation in (4.1) in the main paper. The buy-side entries can be interpreted similarly. In all the four tables, we observe good agreement between the theoretical shapes computed from our limiting model and the empirical shapes of stock BAC on Arca.

Table 2: Order flow statistics of BAC during 12:45pm-12:50pm, August 5, 2010.

i	$\Lambda_A^n(i)$	$\Theta_A^n(i)$	$N_A^n(i)$	$\Lambda_B^n(i)$	$\Theta_B^n(i)$)	$N_B^n(i)$
1	1.394	$0.731 \cdot 10^{-2}$	641	1.429	$0.562 \cdot 10^{-2}$	495
2	0.317	$0.134 \cdot 10^{-2}$	143	0.582	$0.1 \cdot 10^{-2}$	204
3	0.313	$0.129 \cdot 10^{-2}$	146	0.438	$0.151 \cdot 10^{-2}$	154
4	0.313	$0.111 \cdot 10^{-2}$	139	0.476	$0.207 \cdot 10^{-2}$	163
5	0.35	$0.196 \cdot 10^{-2}$	164	0.476	$0.217 \cdot 10^{-2}$	171
6	0.105	$0.039 \cdot 10^{-2}$	44	0.232	$0.11 \cdot 10^{-2}$	77
7	0.025	$0.01 \cdot 10^{-2}$	11	0.005	0	1
8	0.021	$0.012 \cdot 10^{-2}$	8	0.005	$0.004 \cdot 10^{-2}$	2
9	0.017	$0.013 \cdot 10^{-2}$	8	0.005	$0.004 \cdot 10^{-2}$	2
10	0.004	$0.007 \cdot 10^{-2}$	3	0.011	$0.003 \cdot 10^{-2}$	3

Total number of order book events is 2628. $\Upsilon_A^n = \Upsilon_B^n = 0$.

Table 3: Order flow statistics of BAC during 12:45pm-12:55pm, August 5, 2010.

i	$\Lambda_A^n(i)$	$\Theta_A^n(i)$	$N_A^n(i)$	$\Lambda_B^n(i)$	$\Theta_B^n(i)$	$N_B^n(i)$
1	1.012	$0.43 \cdot 10^{-2}$	1104	1.241	$0.545 \cdot 10^{-2}$	1097
2	0.298	$0.132 \cdot 10^{-2}$	332	0.535	$0.094 \cdot 10^{-2}$	445
3	0.535	$0.231 \cdot 10^{-2}$	614	0.400	$0.138 \cdot 10^{-2}$	330
4	0.235	$0.091 \cdot 10^{-2}$	266	0.389	$0.165 \cdot 10^{-2}$	315
5	0.273	$0.15 \cdot 10^{-2}$	310	0.450	$0.201 \cdot 10^{-2}$	374
6	0.054	$0.021 \cdot 10^{-2}$	57	0.233	$0.103 \cdot 10^{-2}$	182
7	0.014	$0.009 \cdot 10^{-2}$	18	0.005	0	2
8	0.016	$0.013 \cdot 10^{-2}$	17	0.005	$0.002 \cdot 10^{-2}$	3
9	0.007	$0.005 \cdot 10^{-2}$	8	0.005	$0.002 \cdot 10^{-2}$	3
10	0.002	$0.003 \cdot 10^{-2}$	3	0.007	$0.001 \cdot 10^{-2}$	4

Total number of order book events is 5565. $\Upsilon_A^n = 0.004, \Upsilon_B^n = 0$.

Table 4: Order flow statistics of BAC during 12:45pm-13:00pm, August 5, 2010.

i	$\Lambda_A^n(i)$	$\Theta_A^n(i)$	$N_A^n(i)$	$\Lambda_B^n(i)$	$\Theta_B^n(i)$	$N_B^n(i)$
1	0.889	$0.31 \cdot 10^{-2}$	1622	1.265	$0.668 \cdot 10^{-2}$	1720
2	0.287	$0.128 \cdot 10^{-2}$	541	0.600	$0.111 \cdot 10^{-2}$	771
3	0.706	$0.303 \cdot 10^{-2}$	1351	0.434	$0.16 \cdot 10^{-2}$	555
4	0.222	$0.085 \cdot 10^{-2}$	422	0.414	$0.184 \cdot 10^{-2}$	522
5	0.254	$0.138 \cdot 10^{-2}$	484	0.493	$0.225 \cdot 10^{-2}$	626
6	0.038	$0.014 \cdot 10^{-2}$	66	0.258	$0.119 \cdot 10^{-2}$	316
7	0.008	$0.005 \cdot 10^{-2}$	19	0.003	0	2
8	0.011	$0.01 \cdot 10^{-2}$	21	0.003	$0.001 \cdot 10^{-2}$	3
9	0.004	$0.003 \cdot 10^{-2}$	8	0.003	$0.001 \cdot 10^{-2}$	3
10	0.001	$0.002 \cdot 10^{-2}$	3	0.005	$0.001 \cdot 10^{-2}$	4

Total number of order book events is 9179. $\Upsilon_A^n = 0.037, \Upsilon_B^n = 0$.

Table 5: Order flow statistics of BAC during 12:45pm-13:05pm, August 5, 2010.

i	$\Lambda_A^n(i)$	$\Theta_A^n(i)$	$N_A^n(i)$	$\Lambda_B^n(i)$	$\Theta_B^n(i)$	$N_B^n(i)$
1	1.16	$0.383 \cdot 10^{-2}$	2702	1.578	$0.787 \cdot 10^{-2}$	2598
2	0.38	$0.177 \cdot 10^{-2}$	934	0.795	$0.152 \cdot 10^{-2}$	1273
3	0.915	$0.408 \cdot 10^{-2}$	2193	0.553	$0.2 \cdot 10^{-2}$	875
4	0.3	$0.112 \cdot 10^{-2}$	689	0.524	$0.211 \cdot 10^{-2}$	805
5	0.353	$0.191 \cdot 10^{-2}$	843	0.609	$0.26 \cdot 10^{-2}$	954
6	0.057	$0.039 \cdot 10^{-2}$	164	0.285	$0.152 \cdot 10^{-2}$	466
7	0.009	$0.01 \cdot 10^{-2}$	35	0.020	0	16
8	0.014	$0.012 \cdot 10^{-2}$	34	0.007	$0.008 \cdot 10^{-2}$	15
9	0.005	$0.003 \cdot 10^{-2}$	10	0.004	$0.002 \cdot 10^{-2}$	6
10	0.002	$0.001 \cdot 10^{-2}$	5	0.010	$0.006 \cdot 10^{-2}$	18

Total number of order book events is 14841. $\Upsilon_A^n = 0.13, \Upsilon_B^n = 0.155$.

Table 6: Comparison of the empirical shape and the theoretical shape for BAC at 12:50pm. Order flow rates are estimated using data in the time interval 12:45pm-12:50pm.

Sell side							
Price bins	\$14–\$14.03	\$14.01–\$14.04	\$14.02–\$14.05	\$14.03–\$14.06	\$14.04–\$14.07	\$14.05–\$14.08	\$14.06–\$14.09
12:50 Arca	806	801	809	780	639	596	497
12:50 Model	826	804	771	733	611	566	499
Error	2.5%	0.3%	-4.6%	-6%	-4.3%	-4.8%	0.3%
Buy side							
Price bins	\$13.96–\$13.99	\$13.95–\$13.98	\$13.94–\$13.97	\$13.93–\$13.96	\$13.92–\$13.95	\$13.91–\$13.94	\$13.90–\$13.93
12:50 Arca	1263	1173	858	692	607	544	564
12:50 Model	1246	1206	835	660	591	521	563
Error	-1.2%	2.9%	-2.6%	-4.5%	-2.7%	-4.2%	-0.1%

Table 7: Comparison of the empirical shape and the theoretical shape for BAC at 12:55pm. Order flow rates are estimated using data in the time interval 12:45pm-12:55pm.

Sell side							
Price bins	\$14–\$14.03	\$14.01–\$14.04	\$14.02–\$14.05	\$14.03–\$14.06	\$14.04–\$14.07	\$14.05–\$14.08	\$14.06–\$14.09
12:55 Arca	856	790	791	792	656	610	515
12:55 Model	880	822	783	735	610	561	493
Error	2.8%	4%	-1%	-7.1%	-7%	-8.1%	-4.2%
Buy side							
Price bins	\$13.96–\$13.99	\$13.95–\$13.98	\$13.94–\$13.97	\$13.93–\$13.96	\$13.92–\$13.95	\$13.91–\$13.94	\$13.90–\$13.93
12:55 Arca	1199	1239	895	726	635	567	567
12:55 Model	1249	1239	882	699	617	543	567
Error	4.3%	0%	1.4%	-3.6%	-2.7%	-4.4%	0%

Table 8: Comparison of the empirical shape and the theoretical shape for BAC at 13:00pm. Order flow rates are estimated using data in the time interval 12:45pm-13:00pm.

Sell side							
Price bins	\$14–\$14.03	\$14.01–\$14.04	\$14.02–\$14.05	\$14.03–\$14.06	\$14.04–\$14.07	\$14.05–\$14.08	\$14.06–\$14.09
13:00 Arca	971	790	792	791	655	609	513
13:00 Model	950	846	804	750	616	561	493
Error	-2.2%	7.2%	1.4%	-5.1%	-6%	-7.9%	-3.9%
Buy side							
Price bins	\$13.96–\$13.99	\$13.95–\$13.98	\$13.94–\$13.97	\$13.93–\$13.96	\$13.92–\$13.95	\$13.91–\$13.94	\$13.90–\$13.93
13:00 Arca	1063	1210	896	747	651	572	567
13:00 Model	1202	1228	889	713	626	551	567
Error	13%	1.4%	-0.8%	-4.7%	-3.8%	-3.6%	0%

Table 9: Comparison of the empirical shape and the theoretical shape for BAC at 13:05pm. Order flow rates are estimated using data in the time interval 12:45pm-13:05pm.

Sell side							
Price bins	\$14–\$14.03	\$14.01–\$14.04	\$14.02–\$14.05	\$14.03–\$14.06	\$14.04–\$14.07	\$14.05–\$14.08	\$14.06–\$14.09
13:05 Arca	930	811	776	762	621	577	518
13:05 Model	981	862	806	751	601	540	489
Error	5.5%	6.3%	3.9%	-1.4%	-3.3%	-6.4%	-5.5%
Buy side							
Price bins	\$13.96–\$13.99	\$13.95–\$13.98	\$13.94–\$13.97	\$13.93–\$13.96	\$13.92–\$13.95	\$13.91–\$13.94	\$13.90–\$13.93
13:05 Arca	1180	1262	945	795	669	581	574
13:05 Model	1239	1272	930	747	629	536	556
Error	5%	0.8%	-1.6%	-5.9%	-6%	-7.7%	-3%

References

[1] Rama Cont, Sasha Stoikov, and Rishi Talreja. A stochastic model for order book dynamics. *Oper. Res.*, 58(3):549–563, 2010.

[2] Xuefeng Gao and Shijie Deng. Hydrodynamic limit of order book dynamics. *Available at* `http://www1.se.cuhk.edu.hk/~xfgao/`.